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On the decoupling of the homogeneous and inhomogeneous parts in inhomogeneous quantum groups

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Abstract
We show that, if there exists a realization of a Hopf algebra H in a H-module algebra $\mathcal{A}$, then one can split their cross-product into the tensor product algebra of $\mathcal{A}$ itself with a subalgebra isomorphic to H and *commuting* with $\mathcal{A}$. This result applies in particular to the algebra underlying inhomogeneous quantum groups like the Euclidean groups, which are obtained as cross-products of the quantum Euclidean spaces $\mathbb{R}_q^N$ with the quantum groups of rotation $U_q so(N)$ of $\mathbb{R}_q^N$, for which it has no classical analogue.

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1. Introduction

As known, given a unital module algebra $\mathcal{A}$ of a Lie algebra $\mathbf{g}$ (over the field $\mathbb{C}$, say), one can build a new module algebra, called cross-product $U\mathbf{g} \triangleright\!\!\!< \mathcal{A}$, that is, as a vector space the tensor product $\mathcal{A} \otimes U\mathbf{g}$ of the vector spaces $\mathcal{A}$, $U\mathbf{g}$ (over the same field) and has the product law

$$(\mathbf{1}_\mathcal{A} \otimes g)(\mathbf{1}_\mathcal{A} \otimes g') = (\mathbf{1}_\mathcal{A} \otimes gg') \tag{1.1}$$

$$(a \otimes \mathbf{1}_H)(a' \otimes \mathbf{1}_H) = aa' \otimes \mathbf{1}_H \tag{1.2}$$

$$(a \otimes \mathbf{1}_H)(\mathbf{1}_\mathcal{A} \otimes g) = a \otimes g \tag{1.3}$$

$$(\mathbf{1}_\mathcal{A} \otimes g)(a \otimes \mathbf{1}_H) = g_{(1)} \triangleright a \otimes g_{(2)} \tag{1.4}$$

for any $g, g' \in U\mathbf{g}$, $a, a' \in \mathcal{A}$. Here we have denoted by $\triangleright$ the left action of the Hopf algebra $H \equiv U\mathbf{g}$ on $\mathcal{A}$,

$$\triangleright : (g, a) \in H \times \mathcal{A} \to g \triangleright a \in \mathcal{A} \tag{1.5}$$

and used a Sweedler-type notation with suppressed summation sign for the co-product $\Delta(g)$ of g, namely the short-hand notation $\Delta(g) = g_{(1)} \otimes g_{(2)}$ instead of a sum $\Delta(g) = \sum_\mu g^\mu_{(1)} \otimes g^\mu_{(2)}$ of many terms. In the main part of this paper we shall work with a left action and therefore left-module algebras. In section 4 we will give the formulae if we use, instead, a right action $\triangleleft$ and right-module algebras. By definition of a (left) action, for any $g, g' \in H$, $a, a' \in \mathcal{A}$,

$$(gg') \triangleright a = g \triangleright (g' \triangleright a) \tag{1.6}$$

$$g \triangleright (aa') = \left(g_{(1)} \triangleright a\right)\left(g_{(2)} \triangleright a'\right). \tag{1.7}$$

We recall that the co-product on the unit and on any $g \in \mathbf{g}$ is given by

$$\Delta(\mathbf{1}_H) = \mathbf{1}_H \otimes \mathbf{1}_H \qquad \Delta(g) = g \otimes \mathbf{1}_H + \mathbf{1}_H \otimes g. \tag{1.8}$$

On the rest of $U\mathbf{g}$ it is determined using the fact that it is an algebra homomorphism $\Delta: H \to H \boldsymbol{\otimes} H$. Clearly, the co-product is co-commutative, i.e. $g_{(1)} \boldsymbol{\otimes} g_{(2)} = g_{(2)} \boldsymbol{\otimes} g_{(1)}$.

For the sake of clarity, above and in the following we denote the tensor products of vector spaces and algebras by $\otimes$ and $\boldsymbol{\otimes}$ (in boldface), respectively; therefore, given two unital algebras B, B' (over the same field), $B \boldsymbol{\otimes} B'$ is $B \otimes B'$ as a vector space, while as an algebra it is characterized by the product

$$(a \otimes a')(b \otimes b') = (ab \otimes a'b') \tag{1.9}$$

for any $a, b \in B$ and $a', b' \in B'$.

In the following, with a standard abuse of notation, for any unital algebras B, B' and any $b \in B$, $b' \in B'$ we shall denote by bb' the element $b \otimes b'$ in the tensor product of vector spaces $B \otimes B'$ and omit $\mathbf{1}_B, \mathbf{1}_{B'}$ whenever multiplied by non-unit elements (thus, $\mathbf{1}_B B'$, $B\mathbf{1}_{B'}$ will be denoted by B' and B). Consequently, in the case of, e.g. the cross-product algebra $U\mathbf{g} \rtimes \mathcal{A}$, relations (1.1)–(1.3) take trivial forms, whereas (1.4) becomes the commutation relation

$$ga = \left(g_{(1)} \triangleright a\right) g_{(2)}. \tag{1.10}$$

For a tensor product algebra $B \boldsymbol{\otimes} B'$ the analogues of relations (1.1)–(1.3) take trivial forms, whereas (1.9) for $a = \mathbf{1}_B$, $b' = \mathbf{1}_{B'}$ becomes the trivial commutation relation

$$a'b = ba'. \tag{1.11}$$

Of course, $B \boldsymbol{\otimes} B'$ is isomorphic to $B' \boldsymbol{\otimes} B$.

$H \rtimes \mathcal{A}$ is itself a module algebra under the left action $\triangleright$ of H if we extend the latter on the elements of the H subalgebra as the adjoint action,

$$g \triangleright h = g_{(1)} h S g_{(2)} \qquad g, h \in H \tag{1.12}$$

(here S denotes the antipode of H), and set as usual

$$g \triangleright (a \otimes h) = g_{(1)} \triangleright a \ \otimes \ g_{(2)} \triangleright h.$$

Note that in the notation mentioned above this relation takes the same form as (1.7), i.e. becomes $g \triangleright (ah) = \left(g_{(1)} \triangleright a\right)\left(g_{(2)} \triangleright h\right)$. It is necessary to show that relation (1.10) implies that one can realize the action $\triangleright: H \times (H \rtimes \mathcal{A}) \to H \rtimes \mathcal{A}$ in the 'adjoint-like way'

$$g \triangleright \eta = g_{(1)} \eta S g_{(2)} \tag{1.13}$$

on all of $H \rtimes \mathcal{A}$.

Classical examples of cross-product algebras are the universal enveloping algebras of inhomogeneous Lie groups, such as the Poincaré algebra, where $H = Uso(3, 1)$ is the algebra generated by infinitesimal Lorentz transformations and $\mathcal{A}$ is the Abelian algebra generated by infinitesimal translations on Minkowski space, or the Euclidean algebra, where $\mathcal{A}$ is the Abelian algebra generated by infinitesimal translations on the N-dimensional Euclidean space $\mathbb{R}^N$ and $H = Uso(N)$ is the algebra generated by its infinitesimal rotations.

The above setting can be 'deformed' by allowing H to be a non-co-commutative Hopf algebra, e.g. the quantum group $U_q\mathbf{g}$, and as $\mathcal{A}$ the corresponding q-deformed module algebra. In general, the latter will no longer be Abelian, even if its classical counterpart is. A cross-product $H{\triangleright\!\!<}\mathcal{A}$ can still be defined by means of the same formulae (1.5)–(1.10).

In this paper, we want to show that there are prominent examples of cross-products $H{\triangleright\!\!<}\mathcal{A}$ that are isomorphic to $\mathcal{A}\otimes H$, more precisely are equal to $\mathcal{A}H'$ with $H' \subset H{\triangleright\!\!<}\mathcal{A}$ a subalgebra isomorphic to H and *commuting* with $\mathcal{A}$, even if this is not the case for their undeformed counterparts. As we shall see, this occurs if there exists an algebra homomorphism φ of the cross-product into $\mathcal{A}$ acting identically on the latter. Of course, this will have dramatic consequences for the cross-product, both from the algebraic and the representation–theoretic viewpoint; it will allow us to reduce representations of $H{\triangleright\!\!<}\mathcal{A}$ to direct sums of tensor products of representations of $\mathcal{A}$ and H'. To prevent misunderstanding, we note that in general if $H{\triangleright\!\!<}\mathcal{A}$ itself is a Hopf algebra, as in the case of inhomogeneous quantum groups, neither $\mathcal{A}$ nor H' will be a Hopf subalgebra of $H{\triangleright\!\!<}\mathcal{A}$.

The present paper has been inspired among others by the results of [3, 8]. In [3] the existence of such a H' for the q-deformed Euclidean algebra in three dimensions has been noted; its generators have been constructed 'by hand' and have been used to decouple q-rotations from q-translations in the $*$-representations of $H{\triangleright\!\!<}\mathcal{A}$. In [8] we had constructed 'by hand' for the q-deformed Euclidean algebra in $N \geqslant 3$ dimensions a set of generators which do the same job; but, instead of commuting with the q-deformed generators of translations, they q-commute with the latter. Now, *a posteriori*, one can check that they can be obtained as products of suitable elements of H' by suitable elements of the natural Cartan (i.e. maximal Abelian) Hopf subalgebra of $H = U_q so(N)$.

In contrast, the present paper gives a very simple prescription for their construction, based on the existence of φ. The prescription can thus be applied to a number of models, including the following. In [2, 3] a class of homomorphisms (2.1) and (2.2) has been determined for a slightly enlarged version $\mathcal{A}$ of the algebra of functions on the N-dimensional quantum Euclidean space $\mathbb{R}_q^N$ or quantum Euclidean sphere S_q^{N-1}, the Hopf algebra H denoting $U_q so(N)$ itself if N is odd, either the Borel subalgebra $U_q^+ so(N)$ or the one $U_q^- so(N)$ if N is even. Their behaviour under the $*$-structures has been investigated in [10] (where incidentally we draw another consequence of its existence, namely the possibility of 'unbraiding' braided tensor product algebras). This will be explicitly described in section 5.1. The analogous maps for the q-deformed fuzzy sphere $S^2_{q,M}$ have been found in [12]. On the other hand, the existence of algebra homomorphisms (2.1) for $H = U_q so(N), U_q sl(N)$ and $\mathcal{A}$ respectively equal to (a suitable completion of) the $U_q so(N)$-covariant Heisenberg algebra or the $U_q sl(N)$-covariant Heisenberg (or Clifford) algebras, has been known for an even longer time [4, 7, 13]. This will be treated in section 5.2. Note that in the latter cases φ also exist for the undeformed counterparts at $q = 1$, thus our results will also apply in this case (we do not know whether this has ever been formulated as a result in ordinary Lie group theory).

In section 2 we state and prove the main results of this work leading to the construction of H'. In section 3 we focus on the $*$-structures. In section 4 we give without proof for right-module algebras all the main formulae valid for left-module algebras. In section 5 we apply our results to the two examples of cross-product algebras mentioned above.

We conclude this section with some additional preliminaries. Beside the Sweedler-type notation with lower indices introduced for the co-product, we shall denote a sum of many terms in a tensor product by a Sweedler-type notation with *upper* indices and suppressed summation sign, e.g. $c^{(1)} \otimes c^{(2)}$ will actually mean a sum $\sum_\mu c_\mu^{(1)} \otimes c_\mu^{(2)}$. Secondly, we can also introduce an 'opposite' action $\triangleright^{op}: H \times (H{\triangleright\!\!<}\mathcal{A}) \to H{\triangleright\!\!<}\mathcal{A}$, i.e. an action of the Hopf algebra with the same algebra structure and co-unit but opposite co-product $\Delta^{op}(g) = g_{(2)} \otimes g_{(1)}$ and inverse antipode by

$$g \triangleright^{op} \eta = g_{(2)} \eta S^{-1} g_{(1)}. \tag{1.14}$$

It fulfils

$$(gg') \triangleright^{op} a = g \triangleright^{op} (g' \triangleright^{op} a) \tag{1.15}$$

$$g \triangleright^{op} (aa') = \left(g_{(2)} \triangleright^{op} a\right)\left(g_{(1)} \triangleright^{op} a'\right). \tag{1.16}$$

(Note that $g \triangleright^{op} a$ in general is *not* an element of $\mathcal{A}$).

2. The commutant of $\mathcal{A}$ within $H \bowtie \mathcal{A}$

2.1. The basic construction

In this section we assume that there exists an algebra homomorphism

$$\varphi : H \bowtie \mathcal{A} \to \mathcal{A} \tag{2.1}$$

acting as the identity on $\mathcal{A}$, namely for any $a \in \mathcal{A}$

$$\varphi(a) = a. \tag{2.2}$$

(Note that, as a consequence, φ is idempotent: $\varphi^2 = \varphi$.) More explicitly, the fact that φ is a homomorphism implies that for any $a \in \mathcal{A}$, $g \in H$

$$\varphi(g)a = \left(g_{(1)} \triangleright a\right) \varphi\left(g_{(2)}\right). \tag{2.3}$$

(Note that (unless the left action of H on $\mathcal{A}$ is trivial), no φ can exist if $\mathcal{A}$ is Abelian, e.g. for the Euclidean algebra $Uso(N) \bowtie \mathbb{R}^N$; but, as we shall recall later, φ exists for the q-deformed Euclidean algebra, and therefore the result will apply.)

Let $\mathcal{C}$ be the commutant of $\mathcal{A}$ within $H \bowtie \mathcal{A}$, i.e. the subalgebra

$$\mathcal{C} := \{c \in H \bowtie \mathcal{A} \mid [c, a] = 0 \quad \forall a \in \mathcal{A}\}. \tag{2.4}$$

Clearly $\mathcal{C}$ contains the centre $\mathcal{Z}(\mathcal{A})$ of $\mathcal{A}$. Let $\zeta : H \to H \bowtie \mathcal{A}$ be the map defined by

$$\zeta(g) := \varphi\left(Sg_{(1)}\right) g_{(2)}. \tag{2.5}$$

Note that if in the definition of ζ we drop φ, we get instead the co-unit ε. Similarly, if we apply φ to ζ and recall that φ is both a homomorphism and idempotent we also find

$$\varphi \circ \zeta = \varepsilon. \tag{2.6}$$

Now, $\varepsilon(g)$ is a complex number and therefore trivially commutes with $\mathcal{A}$. Since φ does not change the commutation relations between $\mathcal{A}$ and H, we also expect that $\zeta(g)$ commutes with $\mathcal{A}$. This is confirmed by

Theorem 1. *Let H be a Hopf algebra, $\mathcal{A}$ a H-module algebra, $\mathcal{C}$ the commutant (2.4) and φ an homomorphism of the type (2.1) and (2.2). Then (2.5) defines an injective algebra homomorphism $\zeta : H \to \mathcal{C}$; moreover, $\mathcal{C} = \mathcal{Z}(\mathcal{A})\zeta(H)$ and $H \bowtie \mathcal{A} = \mathcal{A}\zeta(H)$. If, in particular $\mathcal{Z}(\mathcal{A}) = \mathbb{C}$, then $\mathcal{C} = \zeta(H)$ and $\zeta : H \leftrightarrow \mathcal{C}$ is an algebra isomorphism.*

In other words, the subalgebra H' looked for in the introduction will be obtained by setting $H' := \zeta(H)$. For these reasons we shall call ζ, as well as the other maps ζ_i, $\zeta_i^{\pm}$ which we shall introduce below, *decoupling maps*.

Proof. For any $a \in \mathcal{A}$,

$$\begin{aligned}
\zeta(g)a &\stackrel{(2.5)}{=} \varphi\left(Sg_{(1)}\right) g_{(2)}a \\
&\stackrel{(1.10)}{=} \varphi\left(Sg_{(1)}\right)\left(g_{(2)} \triangleright a\right) g_{(3)} \\
&\stackrel{(2.3)}{=} \left[Sg_{(2)} \triangleright \left(g_{(3)} \triangleright a\right)\right] \varphi\left(Sg_{(1)}\right) g_{(4)} \\
&\stackrel{(1.6)}{=} \left[\left(Sg_{(2)}g_{(3)}\right) \triangleright a\right] \varphi(Sg_{(1)})g_{(4)} \\
&= \left[\varepsilon\left(g_{(2)}\right) \mathbf{1}_H \triangleright a\right] \varphi\left(Sg_{(1)}\right) g_{(3)} \\
&= a\varphi\left(Sg_{(1)}\right) g_{(2)} = a\zeta(g)
\end{aligned} \tag{2.7}$$

proving that $\zeta(g) \in \mathcal{C}$. Using (2.7) with $a = \varphi\big(Sg'_{(1')}\big)$ we find

$$\begin{aligned}
\zeta(gg') &\stackrel{(2.5)}{=} \varphi\left[S\left(g_{(1)}g'_{(1')}\right)\right] g_{(2)}g'_{(2')} = \varphi\left(Sg'_{(1')}\right)\varphi\left(Sg_{(1)}\right) g_{(2)}g'_{(2')} \\
&\stackrel{(2.5)}{=} \varphi\left(Sg'_{(1')}\right)\zeta(g)g'_{(2')} \stackrel{(2.7)}{=} \zeta(g)\varphi\left(Sg'_{(1')}\right) g'_{(2')} \\
&\stackrel{(2.5)}{=} \zeta(g)\zeta(g')
\end{aligned} \tag{2.8}$$

proving that ζ is a homomorphism. To prove that ζ is an injective note that $\zeta(g) = \zeta(g')$ implies

$$\varphi\left(Sg_{(1)}\right) \otimes g_{(2)} = \varphi\left(Sg'_{(1)}\right) \otimes g'_{(2)}$$

whence, by applying $(m \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \varphi \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Delta)$ we find $g = g'$ (we have denoted by m the multiplication map of $\mathcal{A}$, $m(a \otimes b) = ab$).

Now, consider a generic element $c \in H{\triangleright\!\!\!<}\mathcal{A}$ and decompose it in the form $c = c^{(1)}c^{(2)}$ with $c^{(1)} \otimes c^{(2)} \in \mathcal{A} \otimes H$. From (2.5) it immediately follows that

$$c = c^{(1)}c^{(2)} = c^{(1)}\varphi\left(c^{(2)}_{(1)}\right)\zeta\left(c^{(2)}_{(2)}\right)$$

showing that $H{\triangleright\!\!\!<}\mathcal{A} = \mathcal{A}\zeta(H)$ because $c^{(1)}\varphi\big(c^{(2)}_{(1)}\big) \in \mathcal{A}$. In particular, assume $c \in \mathcal{C}$. Then

$$0 = [a, c] = \left[a, c^{(1)}\varphi\left(c^{(2)}_{(1)}\right)\zeta\left(c^{(2)}_{(2)}\right)\right] \stackrel{(2.7)}{=} \left[a, c^{(1)}\varphi\left(c^{(2)}_{(1)}\right)\right]\zeta\left(c^{(2)}_{(2)}\right)$$

since ζ is injective, all factors $\zeta\big(c^{(2)}_{(2)}\big)$ are linearly independent and therefore

$$\left[a, c^{(1)}\varphi\left(c^{(2)}_{(1)}\right)\right] = 0$$

whence we conclude that $c^{(1)}\varphi\big(c^{(2)}_{(1)}\big) \in \mathcal{Z}(\mathcal{A})$ and $c \in \mathcal{Z}(\mathcal{A})\zeta(H)$. □

Corollary 1. *Under the same assumptions as theorem 1 the centre of the cross-product $H{\triangleright\!\!\!<}\mathcal{A}$ is given by*

$$\mathcal{Z}(H{\triangleright\!\!\!<}\mathcal{A}) = \mathcal{Z}(\mathcal{A})\zeta(\mathcal{Z}(H)). \tag{2.9}$$

Moreover, if H_c, $\mathcal{A}_c$ are maximal Abelian subalgebras of H and $\mathcal{A}$, respectively, then $\mathcal{A}_c\zeta(H_c)$ is a maximal Abelian subalgebra of $H{\triangleright\!\!\!<}\mathcal{A}$.

This is almost all that we need in the determination of the Casimirs and a complete set of commuting observables of a quantum system whose algebra of observables is equal to (or contains) $H{\triangleright\!\!\!<}\mathcal{A}$, as in [3, 8]. In addition, we just need that these two subalgebras be closed under the corresponding $*$-structure of $H{\triangleright\!\!\!<}\mathcal{A}$, that will be investigated in section 3.

Proof. The proof of the second statement is immediate. As for the first, if $c \in \mathcal{Z}(H)$ then $[c, H] = 0$ and, applying the homomorphism ζ, $[\zeta(c), \zeta(H)] = 0$; on the other hand,

$[\zeta(c), \mathcal{A}] = 0$ by (2.7). Hence, $\zeta(c)$ commutes with $\mathcal{A}\zeta(H)$, i.e. with $H \triangleright\!\!< \mathcal{A}$, by theorem 1. Similarly if $\tilde{c} \in \mathcal{Z}(\mathcal{A})$ then $[\tilde{c}, \mathcal{A}] = 0$; on the other hand, $[\tilde{c}, \zeta(H)] = 0$ by (2.7). Hence $\tilde{c}$ commutes with $H \triangleright\!\!< \mathcal{A}$. Therefore $\mathcal{Z}(\mathcal{A})\zeta\,(\mathcal{Z}(H)) \subset \mathcal{Z}(H \triangleright\!\!< \mathcal{A})$.

Vice versa, by theorem 1 any $c \in H \triangleright\!\!< \mathcal{A}$ can be expressed in the form $c = c^{(1)}\zeta\left(c^{(2)}\right)$ with $c^{(1)} \otimes c^{(2)} \in \mathcal{A} \otimes H$. If in particular $c \in \mathcal{Z}(H \triangleright\!\!< \mathcal{A})$, then it must be on one hand

$$0 = \left[c^{(1)}\zeta\left(c^{(2)}\right), \mathcal{A}\right] = [c^{(1)}, \mathcal{A}]\zeta\left(c^{(2)}\right)$$

implying $c^{(1)} \in \mathcal{Z}(\mathcal{A})$ by the linear independence of all factors $\zeta\left(c^{(2)}\right)$; on the other hand it must be

$$0 = \left[c^{(1)}\zeta\left(c^{(2)}\right), \zeta(H)\right] = c^{(1)}\left[\zeta\left(c^{(2)}\right), \zeta(H)\right] = c^{(1)}\zeta\left(\left[c^{(2)}, H\right]\right)$$

implying $c^{(2)} \in \mathcal{Z}(H)$, by the linear independence of all factors $c^{(1)}$ and the injectivity of ζ. Therefore $\mathcal{Z}(H \triangleright\!\!< \mathcal{A}) \subset \mathcal{Z}(\mathcal{A})\zeta\,(\mathcal{Z}(H))$. □

Using the results of the theorem we easily show that the restrictions of the left action of H to $\varphi(H)$ and H itself (see (1.12)) look the same:

Proposition 1. *Under the same assumptions as theorem 1, on the images of φ the left action reads*

$$g \triangleright \varphi(h) = \varphi(g \triangleright h) \tag{2.10}$$

equivalently,

$$g\varphi(h) = \varphi\left(g_{(1)} \triangleright h\right) g_{(2)}. \tag{2.11}$$

Proof.

$$\begin{aligned}
\varphi(g \triangleright h) &\overset{(1.12)}{=} \varphi(g_{(1)}hSg_{(2)}) = \varphi(g_{(1)})\varphi(h)\varphi(Sg_{(2)}) \\
&\overset{(2.5)}{=} \varphi\left(g_{(1)}\right)\varphi(h)\zeta\left(g_{(2)}\right)Sg_{(3)} \overset{(2.7)}{=} \varphi\left(g_{(1)}\right)\zeta\left(g_{(2)}\right)\varphi(h)Sg_{(3)} \\
&\overset{(2.5)}{=} g_{(1)}\varphi(h)Sg_{(2)} \overset{(1.12)}{=} g \triangleright \varphi(h).
\end{aligned}$$
□

Apart from $\zeta_1 \equiv \zeta$, other maps fulfilling the same property (2.6) are

$$\begin{aligned}
& & \zeta_2(g) &:= g_{(2)}\varphi\left(S^{-1}g_{(1)}\right) \\
\zeta_3(g) &:= Sg_{(1)}\varphi\left(g_{(2)}\right) & \zeta_4(g) &:= \varphi\left(g_{(2)}\right)S^{-1}g_{(1)} \\
\zeta_5(g) &:= g_{(1)}\varphi\left(Sg_{(2)}\right) & \zeta_6(g) &:= \varphi\left(S^{-1}g_{(2)}\right)g_{(1)} \\
\zeta_7(g) &:= S^{-1}g_{(2)}\varphi\left(g_{(1)}\right) & \zeta_8(g) &:= \varphi\left(g_{(1)}\right)Sg_{(2)}.
\end{aligned} \tag{2.12}$$

One could wonder whether they also fulfil the previous theorems. Using (2.11) one can easily show that

- $$\zeta_2 = \zeta \tag{2.13}$$
- $\zeta_3 = \zeta \circ S$, $\zeta_4 = \zeta_2 \circ S^{-1} = \zeta \circ S^{-1}$ so that $\zeta_3(g), \zeta_4(g) \in \mathcal{C}$, but ζ_3, ζ_4 are antihomomorphisms;
- $\zeta_5, \zeta_6, \zeta_7, \zeta_8$ do *not* map H into $\mathcal{C}$.

Let us prove for instance the first statement:

$$\begin{aligned}
\zeta_2(g) &\overset{(2.12)}{=} g_{(2)}\varphi\left(S^{-1}g_{(1)}\right) \overset{(2.11)}{=} \varphi\left(g_{(2)} \triangleright S^{-1}g_{(1)}\right)g_{(3)} \\
&\overset{(1.12)}{=} \varphi\left(g_{(2)}S^{-1}g_{(1)}Sg_{(3)}\right)g_{(4)} = \varphi\left(Sg_{(1)}\right)g_{(2)} \overset{(2.5)}{=} \zeta(g).
\end{aligned}$$

How does $\zeta(H)$ transform under the action $\triangleright$ of H? One can easily verify that it is not mapped into itself. On the contrary, under the opposite action $\triangleright^{op}$ it is

Proposition 2. *Under the same assumptions as theorem 1, for any $g, h \in H$*

$$h \triangleright^{op} \zeta(g) = \zeta(h \triangleright^{op} g) \qquad (2.14)$$

$$h\zeta(g) = \zeta\left(h_{(2)} \triangleright^{op} g\right) h_{(1)}. \qquad (2.15)$$

Proof. By (1.13), (2.15) is a direct consequence of (2.14). To prove the latter,

$$\begin{aligned}
\zeta(h \triangleright^{op} g) &\stackrel{(1.14)}{=} \zeta\left(h_{(2)} g S^{-1} h_{(1)}\right) \stackrel{(2.8)}{=} \zeta\left(h_{(2)}\right) \zeta(g) \zeta\left(S^{-1} h_{(1)}\right) \\
&\stackrel{(2.5),(2.13)}{=} \zeta_2\left(h_{(3)}\right) \zeta(g) \varphi\left(h_{(2)}\right) S^{-1} h_{(1)} \\
&\stackrel{(2.7)}{=} \zeta_2\left(h_{(3)}\right) \varphi\left(h_{(2)}\right) \zeta(g) S^{-1} h_{(1)} \\
&\stackrel{(2.12)}{=} h_{(4)} \varphi\left(S^{-1} h_{(3)}\right) \varphi\left(h_{(2)}\right) \zeta(g) S^{-1} h_{(1)} \\
&= h_{(2)} \zeta(g) S^{-1} h_{(1)} \stackrel{(1.14)}{=} h \triangleright^{op} \zeta(g).
\end{aligned}$$

□

2.2. Construction adapted to Gauss decompositions of H

In view of the applications that we shall consider in section 5 it is now useful to consider the case that, instead of a φ we just have at our disposal two homomorphisms φ^+, φ^-

$$\varphi^{\pm}: H^{\pm} \triangleright\!\!< \mathcal{A} \rightarrow \mathcal{A} \qquad (2.16)$$

fulfilling (2.2), where H^+, H^- denote two Hopf subalgebras of H such that Gauss decompositions $H = H^+H^- = H^-H^+$ hold. (The typical case is when $H = U_q\mathbf{g}$ and H^+, H^- denote its positive and negative Borel subalgebras.) Then the theorems listed so far will apply separately to $H^+ \triangleright\!\!< \mathcal{A}$ and $H^- \triangleright\!\!< \mathcal{A}$, if we define the corresponding maps $\zeta^{\pm}: H^{\pm} \rightarrow \mathcal{A}$ by

$$\zeta^{\pm}(g) := \varphi^{\pm}\left(Sg_{(1)}\right) g_{(2)} \qquad (2.17)$$

where $g \in H^{\pm}$ respectively. What can we say about the whole $H \triangleright\!\!< \mathcal{A}$? We now prove

Theorem 2. *Let H be a Hopf algebra, $\mathcal{A}$ a H-module algebra, $\mathcal{C}$ the commutant (2.4) and $\varphi^{\pm}$ homomorphisms of the type (2.16) and (2.2). Under the above assumptions formulae (2.17) define injective algebra homomorphisms $\zeta^{\pm}: H^{\pm} \rightarrow \mathcal{C}$. Moreover,*

$$\mathcal{C} = \mathcal{Z}(\mathcal{A})\zeta^+(H^+)\zeta^-(H^-) = \mathcal{Z}(\mathcal{A})\zeta^-(H^-)\zeta^+(H^+) \qquad (2.18)$$

and

$$H \triangleright\!\!< \mathcal{A} = \mathcal{A}\zeta^+(H^+)\zeta^-(H^-) = \mathcal{A}\zeta^-(H^-)\zeta^+(H^+). \qquad (2.19)$$

In particular, if $\mathcal{Z}(\mathcal{A}) = \mathbb{C}$, then $\mathcal{C} = \zeta^+(H^+)\zeta^-(H^-) = \zeta^-(H^-)\zeta^+(H^+)$.

Proof. As anticipated, the fact that $\zeta^{\pm}$ are injective algebra homomorphisms $\zeta^{\pm}: H^{\pm} \rightarrow \mathcal{C}$ follows from theorem 1. Now, by the Gauss decomposition $H = H^+H^-$ a generic element $c \in H \triangleright\!\!< \mathcal{A}$ can be decomposed in the form $c = c^{(1)}c^{(2)}c^{(3)}$ with $c^{(1)} \otimes c^{(2)} \otimes c^{(3)} \in \mathcal{A} \otimes H^+ \otimes H^-$; again, the Sweedler-type notation at the rhs means that a sum of many terms is understood. From (2.17) it immediately follows that

$$\begin{aligned}
c = {} & c^{(1)}c^{(2)}c^{(3)} \\
= {} & c^{(1)}\varphi^+\left(c^{(2)}_{(1)}\right)\zeta^+\left(c^{(2)}_{(2)}\right)\varphi^-\left(c^{(3)}_{(1')}\right)\zeta^-\left(c^{(3)}_{(2')}\right) \\
\stackrel{(2.7)}{=} {} & \zeta^+\left(c^{(2)}_{(2)}\right)\zeta^-\left(c^{(3)}_{(2')}\right)c^{(1)}\varphi^+\left(c^{(2)}_{(1)}\right)\varphi^-\left(c^{(3)}_{(1')}\right)
\end{aligned}$$

showing that $H{\triangleright\!\!\!<}\mathcal{A} = \mathcal{A}\zeta^+(H^+)\zeta^-(H^-)$. In particular, assume $c \in \mathcal{C}$. Then

$$\begin{aligned} 0 = \quad [a, c] &= \left[a, c^{(1)}\varphi^+\left(c^{(2)}_{(1)}\right)\varphi^-\left(c^{(3)}_{(1')}\right)\zeta^+\left(c^{(2)}_{(2)}\right)\zeta^-\left(c^{(3)}_{(2')}\right)\right] \\ &\stackrel{(2.7)}{=} \left[a, c^{(1)}\varphi^+\left(c^{(2)}_{(1)}\right)\varphi^-\left(c^{(3)}_{(1')}\right)\right]\zeta^+\left(c^{(2)}_{(2)}\right)\zeta^-\left(c^{(3)}_{(2')}\right) \end{aligned}$$

since ζ^+, ζ^- are injective, all factors $\zeta^+\big(c^{(2)}_{(2)}\big)\zeta^-\big(c^{(3)}_{(2')}\big)$ are linearly independent and therefore

$$\left[a, c^{(1)}\varphi^+\left(c^{(2)}_{(1)}\right)\varphi^-\left(c^{(3)}_{(1')}\right)\right] = 0$$

whence we conclude that $c^{(1)}\varphi^+\big(c^{(2)}_{(1)}\big)\varphi^-\left(c^{(3)}_{(1')}\right) \in \mathcal{Z}(\mathcal{A})$ and c belongs to $\mathcal{Z}(\mathcal{A})\zeta^+(H^+)\zeta^-(H^-)$.

The proof of the claims with $\zeta^+(H^+), \zeta^-(H^-)$ in the inverse order follows in the same way from the Gauss decomposition $H = H^-H^+$. □

As a consequence of this theorem, for any $g^+ \in H^+$, $g^- \in H^-$ there exists a sum $c^{(1)} \otimes c^{(2)} \otimes c^{(3)} \in \mathcal{Z}(\mathcal{A}) \otimes H^- \otimes H^+$ (depending on g^+, g^-) such that

$$\zeta^+(g^+)\zeta^-(g^-) = c^{(1)}\zeta^-\left(c^{(2)}\right)\zeta^+\left(c^{(3)}\right). \tag{2.20}$$

These will be the 'commutation relations' between elements of $\zeta^+(H^+)$ and $\zeta^-(H^-)$. Their form will depend on the specific algebras considered. In section 5 we shall determine these commutation relations for two examples of cross-products with $H = U_q\mathbf{g}$ using the Faddeev–Reshetikhin–Takhtadjan generators of $U_q\mathbf{g}$.

Of course, propositions 1 and 2 will still apply to the present situation if both g, h belong to H^+ and we replace φ, ζ by φ^+, ζ^+ (or the same with + replaced by −). What can we say otherwise?

Proposition 3. *Under the same assumptions as theorem 2.2, if $g \in H^\pm$, $h \in H^\mp$*

$$g \triangleright \varphi^\mp(h) = \varphi^\pm\left(g_{(1)}\right)\varphi^\mp(h)\varphi^\pm\left(Sg_{(2)}\right) \tag{2.21}$$

$$g\varphi^\mp(h) = \varphi^\pm\left(g_{(1)}\right)\varphi^\mp(h)\varphi^\pm\left(Sg_{(2)}\right)g_{(3)} \tag{2.22}$$

$$g \triangleright^{op} \zeta^\mp(h) = \zeta^\pm\left(g_{(2)}\right)\zeta^\mp(h)\zeta^\pm\left(S^{-1}g_{(1)}\right) \tag{2.23}$$

$$g\zeta^\mp(h) = \zeta^\pm\left(g_{(3)}\right)\zeta^\mp(h)\zeta^\pm\left(S^{-1}g_{(2)}\right)g_{(1)}. \tag{2.24}$$

The proof uses theorem 2.2 and is similar to the proofs of propositions 1 and 2. Similarly, the analogue of corollary 1 reads as

Corollary 2. *Under the same assumptions as theorem 2.2, any element c of the centre $\mathcal{Z}(H{\triangleright\!\!\!<}\mathcal{A})$ of the cross-product $H{\triangleright\!\!\!<}\mathcal{A}$ can be expressed in the form*

$$c = \zeta^+\left(c^{(1)}\right)\zeta^-\left(c^{(2)}\right)c^{(3)} \tag{2.25}$$

where $c^{(1)} \otimes c^{(2)} \otimes c^{(3)} \in H^+ \otimes H^- \otimes \mathcal{Z}(\mathcal{A})$ and $c^{(1)}c^{(2)} \otimes c^{(3)} \in \mathcal{Z}(H) \otimes \mathcal{Z}(\mathcal{A})$; alternatively any such object c is an element of $\mathcal{Z}(H{\triangleright\!\!\!<}\mathcal{A})$. If $H_c \subset H^+ \cap H^-$ and $\mathcal{A}_c$ are maximal Abelian subalgebras of H and $\mathcal{A}$, respectively, then $\mathcal{A}_c\zeta^+(H_c)$ (as well as $\mathcal{A}_c\zeta^-(H_c)$) is a maximal Abelian subalgebra of $H{\triangleright\!\!\!<}\mathcal{A}$.

3. ∗-structures

Assume that H is a Hopf ∗-algebra and $\mathcal{A}$ a H-module ∗-algebra, which means that on H and $\mathcal{A}$ there exist antilinear involutive antihomomorphisms, both of which we denote by the symbol $*$, such that

$$[\varepsilon(g)]^* = \varepsilon(g^*) \tag{3.1}$$

$$(g^*)_{(1)} \otimes (g^*)_{(2)} = \left(g_{(1)}\right)^* \otimes \left(g_{(2)}\right)^* \tag{3.2}$$

$$(Sg)^* = S^{-1}g^* \tag{3.3}$$

$$(g \triangleright a_i)^* = (S^{-1}g^*) \triangleright a_i^* \tag{3.4}$$

(actually (3.3) is a consequence of (3.1) and (3.2) and the uniqueness of the antipode). Then these two $*$-structures can be glued in a unique way to make $H \bowtie \mathcal{A}$ a $*$-algebra. If

$$\varphi \colon H \bowtie \mathcal{A} \to \mathcal{A} \tag{3.5}$$

is a homomorphism acting as the identity on $\mathcal{A}$, then it is straightforward to check that $\varphi' := * \circ \varphi \circ *$ also is. As a consequence, if in a concrete case we know that such a homomorphism is unique, then $\varphi' = \varphi$ and we automatically conclude that it is a $*$-homomorphism,

$$\varphi(\alpha^*) = [\varphi(\alpha)]^* \qquad \alpha \in H \bowtie \mathcal{A}. \tag{3.6}$$

More generally, if the homomorphism φ is not unique, it is natural to look for one that is a $*$-homomorphism. We will give explicit examples of this in section 5.

Proposition 4. *If $\varphi \colon H \bowtie \mathcal{A} \to \mathcal{A}$ is a $*$-homomorphism, then the map $\zeta \colon H \to \mathcal{C}$ also is.*

Proof.

$$\begin{aligned}
\zeta(g^*) &\overset{(2.5)}{=} \varphi\left(Sg^*_{(1)}\right) g^*_{(2)} \overset{(3.3),(3.2)}{=} \varphi\left(\left(S^{-1}g_{(1)}\right)^*\right) g^*_{(2)} \\
&\overset{(3.6)}{=} \left[\varphi\left(S^{-1}g_{(1)}\right)\right]^* g^*_{(2)} = \left[g_{(2)}\varphi\left(S^{-1}g_{(1)}\right)\right]^* \\
&\overset{(2.12)}{=} [\zeta_2(g)]^* \overset{(2.13)}{=} [\zeta(g)]^*.
\end{aligned}$$

□

Alternatively, it may happen that no $*$-homomorphism φ exists, but there exist homomorphisms $\varphi^\pm$ of the type (2.16). If $H^\pm$ are Hopf $*$-subalgebras (i.e. are closed under $*$), then $\varphi'^\pm := * \circ \varphi^\pm \circ *$ are also homomorphisms of the type (2.16), and as before we can look for $\varphi^\pm$ that are $*$-homomorphisms,

$$\varphi^\pm(\alpha^*) = [\varphi^\pm(\alpha)]^* \qquad \alpha \in H^\pm \bowtie \mathcal{A}. \tag{3.7}$$

In contrast, if $H^\pm$ are mapped into each other by $*$, then $* \circ \varphi^\pm \circ *$ is a homomorphism of the type $\varphi^\mp$, and we can look for one such that

$$\varphi^\pm(\alpha^*) = [\varphi^\mp(\alpha)]^* \qquad \alpha \in H^\mp \bowtie \mathcal{A}. \tag{3.8}$$

Proposition 5. *If $\varphi^\pm$ are $*$-homomorphisms, then the map $\zeta^\pm \colon H^\pm \to \mathcal{C}$ also is. If $\varphi^\pm$ fulfil (3.8), then $\zeta^\pm$ fulfil*

$$\zeta^\pm(g^*) = [\zeta^\mp(g)]^* \qquad g \in H^\mp. \tag{3.9}$$

Proof. The first statement amounts to the preceding proposition applied to $H^\pm \bowtie \mathcal{A}$. As for the second, the proof is just a small variation:

$$\begin{aligned}
\zeta^\pm(g^*) &\overset{(2.5)}{=} \varphi^\pm\left(Sg^*_{(1)}\right) g^*_{(2)} \overset{(3.3),(3.2)}{=} \varphi^\pm\left(\left(S^{-1}g_{(1)}\right)^*\right) g^*_{(2)} \\
&\overset{(3.8)}{=} \left[\varphi^\mp\left(S^{-1}g_{(1)}\right)\right]^* g^*_{(2)} = \left[g_{(2)}\varphi^\mp\left(S^{-1}g_{(1)}\right)\right]^* \\
&\overset{(2.12)}{=} \left[\zeta_2^\mp(g)\right]^* \overset{(2.13)}{=} [\zeta^\mp(g)]^*.
\end{aligned}$$

□

4. Formulae for right-module algebras

In this section, we give the analogues for right H-module algebras of the main results found so far for left H-module algebras. By definition the right action $\triangleleft: \mathcal{A} \times H \to \mathcal{A}$ fulfils

$$a \triangleleft (gg') = (a \triangleleft g) \triangleleft g' \tag{4.1}$$

$$(aa') \triangleleft g = \left(a \triangleleft g_{(1)}\right)\left(a' \triangleleft g_{(2)}\right). \tag{4.2}$$

The algebra $\mathcal{A} \rtimes H$ is defined as follows. As a vector space it is $H \otimes \mathcal{A}$, whereas the product is defined through formulae obtained from (1.1)–(1.3) by flipping the tensor factors, together with

$$(\mathbf{1}_H \otimes a)(g \otimes \mathbf{1}_{\mathcal{A}}) = g_{(1)} \otimes \left(a \triangleleft g_{(2)}\right)$$

for any $a \in \mathcal{A}$, $g \in H$. As before, we shall denote $g \otimes a$ by ga and omit unit $\mathbf{1}_{\mathcal{A}}, \mathbf{1}_H$ whenever multiplied by non-unit elements; consequently only the last condition takes a non-trivial form and becomes

$$ag = g_{(1)}\left(a \triangleleft g_{(2)}\right). \tag{4.3}$$

$\mathcal{A} \rtimes H$ itself is a H-module algebra under the right action $\triangleleft$ of H if we extend the latter on the elements of H as the adjoint action,

$$h \triangleleft g = Sg_{(1)} h g_{(2)} \qquad g, h \in H. \tag{4.4}$$

Let $\tilde{\mathcal{C}}$ be the commutant of $\mathcal{A}$ within $\mathcal{A} \rtimes H$. Clearly, it contains the centre $\mathcal{Z}(\mathcal{A})$ of $\mathcal{A}$. We shall need homomorphisms

$$\tilde{\varphi}: \mathcal{A} \rtimes H \to \mathcal{A} \tag{4.5}$$

acting as the identity on $\mathcal{A}$, namely for any $a \in \mathcal{A}$

$$\tilde{\varphi}(a) = a. \tag{4.6}$$

Theorem 3. *Let H be a Hopf algebra, $\mathcal{A}$ a right H-module algebra, $\tilde{\mathcal{C}}$ the commutant of $\mathcal{A}$ within $\mathcal{A} \rtimes H$ and $\tilde{\varphi}$ a homomorphism of the type (4.5), (4.6). Then the map ζ_5 defined by (2.12) (with φ replaced by $\tilde{\varphi}$) is an injective algebra homomorphism $\zeta_5: H \to \tilde{\mathcal{C}}$; moreover, $\tilde{\mathcal{C}} = \zeta_5(H)\mathcal{Z}(\mathcal{A})$ and $\mathcal{A} \rtimes H = \zeta_5(H)\mathcal{A}$. If, in particular, $\mathcal{Z}(\mathcal{A}) = \mathbb{C}$, then $\tilde{\mathcal{C}} = \zeta(H)$ and $\zeta_5: H \leftrightarrow \tilde{\mathcal{C}}$ is an algebra isomorphism.*

Consider the maps ζ_i defined by (2.12), but with φ replaced by $\tilde{\varphi}$. One can easily show that

- $$\zeta_6 = \zeta_5 \tag{4.7}$$
- $\zeta_8 = \zeta_6 \circ S = \zeta_5 \circ S$, $\zeta_7 = \zeta_5 \circ S^{-1}$, so that $\zeta_7(g), \zeta_8(g) \in \mathcal{C}$, but ζ_7, ζ_8 are antihomomorphisms;
- $\zeta_1, \zeta_2, \zeta_3, \zeta_4$ do *not* map H into $\tilde{\mathcal{C}}$.

Proposition 6. *If $g, h \in H$*

$$\tilde{\varphi}(h) \triangleleft g = \tilde{\varphi}(h \triangleleft g) \tag{4.8}$$

$$\tilde{\varphi}(h) g = g_{(1)} \tilde{\varphi}\left(h \triangleleft g_{(2)}\right). \tag{4.9}$$

If, instead of a $\tilde{\varphi}$, we just have at our disposal two homomorphisms $\tilde{\varphi}^+$, $\tilde{\varphi}^-$

$$\tilde{\varphi}^{\pm}: \mathcal{A} \bowtie H^{\pm} \to \mathcal{A} \tag{4.10}$$

where H^+, H^- denote two Hopf subalgebras of H such that the Gauss decompositions $H = H^+H^- = H^-H^+$ hold, then the theorems listed so far will apply separately to $\mathcal{A} \bowtie H^+$ and $\mathcal{A} \bowtie H^-$, if we define corresponding maps $\zeta_5^{\pm}: H^{\pm} \to \mathcal{A}$ by

$$\zeta_5^{\pm}(g) := g_{(1)}\tilde{\varphi}^{\pm}\left(Sg_{(2)}\right) \tag{4.11}$$

where $g \in H^{\pm}$ respectively. More precisely:

Theorem 4. *Under the above assumptions formulae (4.11) define injective algebra homomorphisms $\zeta_5^{\pm}: H^{\pm} \to \tilde{\mathcal{C}}$. Moreover,*

$$\tilde{\mathcal{C}} = \zeta_5^+(H^+)\zeta_5^-(H^-)\mathcal{Z}(\mathcal{A}) = \zeta_5^-(H^-)\zeta_5^+(H^+)\mathcal{Z}(\mathcal{A}) \tag{4.12}$$

and

$$\mathcal{A} \bowtie H = \zeta_5^+(H^+)\zeta_5^-(H^-)\mathcal{A} = \zeta_5^-(H^-)\zeta_5^+(H^+)\mathcal{A}. \tag{4.13}$$

In particular, if $\mathcal{Z}(\mathcal{A}) = \mathbb{C}$, then $\tilde{\mathcal{C}} = \zeta_5^+(H^+)\zeta_5^-(H^-) = \zeta_5^-(H^-)\zeta_5^+(H^+)$.

Proposition 7. *If $g \in H^{\pm}, h \in H^{\mp}$*

$$\tilde{\varphi}^{\mp}(h) \triangleleft g = \tilde{\varphi}^{\pm}\left(Sg_{(1)}\right)\tilde{\varphi}^{\mp}(h)\tilde{\varphi}^{\pm}\left(g_{(2)}\right) \tag{4.14}$$

$$\tilde{\varphi}^{\mp}(h)g = g_{(1)}\tilde{\varphi}^{\pm}\left(Sg_{(2)}\right)\tilde{\varphi}^{\mp}(h)\tilde{\varphi}^{\pm}\left(g_{(3)}\right). \tag{4.15}$$

If H is a Hopf $*$-algebra and $\mathcal{A}$ a H-module $*$-algebra, then the two $*$-structures of H and $\mathcal{A}$ can be glued into a unique structure to make $\mathcal{A} \bowtie H$ a $*$-algebra itself.

Proposition 8. *If $\tilde{\varphi}: \mathcal{A} \bowtie H \to \mathcal{A}$ is a $*$-homomorphism, then the map $\zeta_5: H \to \tilde{\mathcal{C}}$ also is.*

Proposition 9. *If the maps $\tilde{\varphi}^{\pm}$ defined in (4.11) are $*$-homomorphisms, then the maps $\zeta_5^{\pm}: H^{\pm} \to \mathcal{C}$ also are. If $\tilde{\varphi}^{\pm}$ fulfil*

$$\tilde{\varphi}^{\pm}(\alpha^*) = [\tilde{\varphi}^{\mp}(\alpha)]^* \qquad \alpha \in \mathcal{A} \bowtie H^{\mp} \tag{4.16}$$

then $\zeta_5^{\pm}$ fulfil

$$\zeta_5^{\pm}(g^*) = \left[\zeta_5^{\mp}(g)\right]^* \qquad g \in H^{\mp}. \tag{4.17}$$

5. Applications

We now consider a couple of applications where H is the quantum group $U_q\mathbf{g}$ [5], with $\mathbf{g} = sl(N)$ or $\mathbf{g} = so(N)$. As a set of generators of $U_q\mathbf{g}$ it is convenient to introduce the FRT generators [6] $\mathcal{L}^{+i}{}_j, \mathcal{L}^{-i}{}_j$ (i, j take N different values), together with the square roots of the diagonal elements $\mathcal{L}^{+j}{}_j, \mathcal{L}^{-j}{}_j$. In the appendix we recall the relations they fulfil. The FRT generators are related to the so-called universal R-matrix $\mathcal{R}$ by

$$\mathcal{L}^{+a}{}_l := \mathcal{R}^{(1)}\rho^a_l\left(\mathcal{R}^{(2)}\right) \qquad \mathcal{L}^{-a}{}_l := \rho^a_l\left(\mathcal{R}^{-1(1)}\right)\mathcal{R}^{-1(2)} \tag{5.1}$$

where we have denoted by ρ the fundamental N-dimensional representation of $U_qsl(N)$ or $U_qso(N)$. Since in our conventions $\mathcal{R} \in H^+ \otimes H^-$ (H^+, H^- denote the positive, negative Borel subalgebras) we see that $\mathcal{L}^{+a}{}_l \in H^+$ and $\mathcal{L}^{-a}{}_l \in H^-$.

For historical reasons we introduce algebras $\mathcal{A}$ as right- (rather than left-) $U_q\mathbf{g}$-module algebras.

5.1. The Euclidean quantum group $\mathbb{R}_q^N \rtimes U_q so(N)$

As algebra $\mathcal{A}$ we shall consider a slight extension of the quantum Euclidean space $\mathbb{R}_q^N$ [6] (the $U_q so(N)$-covariant quantum space), i.e. the unital associative algebra generated by p^i fulfilling the relations

$$\mathcal{P}_{a\,hk}^{\;ij}\, p^h p^k = 0 \tag{5.2}$$

where $\mathcal{P}_a$ denotes the q-deformed antisymmetric projector appearing in the decomposition of the braid matrix $\hat{R}$ of $U_q so(N)$ (given in formula (A.10)); the latter is related to $\mathcal{R}$ by $\hat{R}^{ij}_{hk} = \rho^j_h\big(\mathcal{L}^{+i}{}_k\big) = \big(\rho^j_h \otimes \rho^i_k\big)(\mathcal{R})$. The multiplet (p^i) carries the fundamental N-dim (or vector) representation ρ of $U_q so(N)$: for any $g \in U_q so(N)$

$$p^i \triangleleft g = \rho^i_j(g)\, p^j. \tag{5.3}$$

This implies

$$p^i \mathcal{L}^{\pm a}{}_b = \mathcal{L}^{\pm a}{}_c\, p^j \hat{R}^{\pm 1\, ci}{}_{jb} \tag{5.4}$$

$$S\mathcal{L}^{\pm a}{}_b\, p^i = \hat{R}^{\pm 1\, ai}{}_{jk}\, p^j\, S\mathcal{L}^{\pm k}{}_b. \tag{5.5}$$

To define $\tilde{\varphi}$ or $\tilde{\varphi}^{\pm}$ one [2] slightly enlarges $\mathbb{R}_q^N$ as follows. One introduces some new generators $\sqrt{P_a}$, with $1 \leqslant a \leqslant \frac{N}{2}$, together with their inverses $(\sqrt{P_a})^{-1}$ requiring that

$$P_a^2 = \sum_{h=-a}^{a} p^h p_h = \sum_{h,k=-a}^{a} g_{hk}\, p^h p^k. \tag{5.6}$$

In the previous equation g_{hk} denotes the 'metric matrix' of $SO_q(N)$:

$$g_{ij} = g^{ij} = q^{-\rho_i}\delta_{i,-j}. \tag{5.7}$$

It is a $SO_q(N)$-isotropic tensor and is a deformation of the ordinary Euclidean metric. Here and in the following $n := \left[\frac{N}{2}\right]$ is the rank of $so(N)$, the indices take the values $i = -n, \ldots, -1, 0, 1, \ldots, n$ for N odd, and $i = -n, \ldots, -1, 1, \ldots, n$ for N even. We have also introduced the notation $(\rho_i) = \big(n - \frac{1}{2}, \ldots, \frac{1}{2}, 0, -\frac{1}{2}, \ldots, \frac{1}{2} - n\big)$ for N odd, $(n-1, \ldots, 0, 0, \ldots, 1-n)$ for N even. In the sequel we shall call P_n^2 also P^2. Moreover, for odd N we also add $\sqrt{p^0}$ and its inverse as new generators. The commutation relations involving these new generators can be fixed consistently and turn out to be simply q-commutation relations. P plays the role of 'deformed Euclidean distance' of the generic 'point of coordinates' (p^i) of $\mathbb{R}_q^N$ from the 'origin'; P_a is the 'projection' of P on the 'subspace' $p^i = 0$, $|i| > a$. The centre of $\mathbb{R}_q^N$ is generated by $\sqrt{P}$ and, only in the case of even N, by $\sqrt{\frac{p^1}{p^{-1}}}$ and its inverse $\sqrt{\frac{p^{-1}}{p^1}}$. In the case of even N one also needs to include the FRT generator $\mathcal{L}^{-1}{}_1 = \mathcal{L}^{+-1}{}_{-1}$ and its inverse $\mathcal{L}^{+1}{}_1 = \mathcal{L}^{--1}{}_{-1}$ (which are generators of $U_q so(N)$ belonging to the natural Cartan subalgebra) among the generators of $\mathcal{A}$. From (5.4) one derives that they satisfy the commutation relations

$$\mathcal{L}^{-1}{}_1 p^{\pm 1} = q^{\pm 1} p^{\pm 1} \mathcal{L}^{-1}{}_1 \qquad \mathcal{L}^{-1}{}_1 p^{\pm i} = p^{\pm i} \mathcal{L}^{-1}{}_1 \qquad \text{for} \quad i > 1 \tag{5.8}$$

with the generators of $\mathcal{A}$, and the standard FRT relations with the rest of $U_q so(N)$. As a consequence, $\sqrt{\frac{p^{\pm 1}}{p^{\mp 1}}}$ are eliminated from the centre of $\mathcal{A}$ (in fact $\mathcal{L}^{\pm 1}{}_1$ do not q-commute with $\sqrt{\frac{p^{\pm 1}}{p^{\mp 1}}}$).

One can easily show that the extension of the action of $U_q so(N)$ to $\sqrt{P_a}$, $(\sqrt{P_a})^{-1}$ is uniquely determined by the constraints that the latter fulfil; it is a bit complicated and therefore will be omitted, since we will not need its explicit expression. We keep the action of

H on $\mathcal{L}^{-1}{}_1$ as the standard (right) adjoint action (4.4). Note that the maps $\tilde{\varphi}^{\pm}$ have no analogue in the 'undeformed' case ($q = 1$) because $\mathcal{A}_1 \equiv \mathbb{R}^N$ is Abelian whereas $H \equiv U_q so(N)$ is not.

The homomorphisms [2] $\tilde{\varphi}^{\pm}: \mathcal{A} \rtimes U_q^{\pm} so(N) \to \mathcal{A}$ take the simplest and most compact expression on the FRT generators of $U_q^{\pm} so(N)$. Let us introduce the short-hand notation $[A, B]_x = AB - xBA$. The images of $\tilde{\varphi}^-$ on the negative FRT generators read as

$$\tilde{\varphi}^- \left(\mathcal{L}^{-i}{}_j\right) = g^{ih} \left[\mu_h, p^k\right]_q g_{kj} \tag{5.9}$$

where

$$\begin{aligned} &\mu_0 = \gamma_0 (p^0)^{-1} && \text{for} \quad N \text{ odd} \\ &\mu_{\pm 1} = \gamma_{\pm 1} (p^{\pm 1})^{-1} \mathcal{L}^{\pm 1}{}_1 && \text{for} \quad N \text{ even} \\ &\mu_a = \gamma_a P_{|a|}^{-1} P_{|a|-1}^{-1} p^{-a} && \text{otherwise} \end{aligned} \tag{5.10}$$

and $\gamma_a \in \mathbb{C}$ are normalization constants fulfilling the conditions

$$\begin{aligned} &\gamma_0 = -q^{-\frac{1}{2}} h^{-1} && \text{for} \quad N \text{ odd} \\ &\gamma_{\pm 1} = -k^{-1} && \text{for} \quad N \text{ even} \\ &\gamma_1 \gamma_{-1} = -q^{-1} h^{-2} && \text{for} \quad N \text{ odd} \\ &\gamma_a \gamma_{-a} = -q^{-1} k^{-2} \omega_a \omega_{a-1} && \text{for} \quad a > 1. \end{aligned}$$

Here $h := q^{\frac{1}{2}} - q^{-\frac{1}{2}}$, $k := q - q^{-1}$, $\omega_a := (q^{\rho_a} + q^{-\rho_a})$. On the other hand, the images of $\tilde{\varphi}^+$ on the positive FRT generators read as

$$\tilde{\varphi}^+ \left(\mathcal{L}^{+i}{}_j\right) = g^{ih} \left[\bar{\mu}_h, p^k\right]_{q^{-1}} g_{kj} \tag{5.12}$$

where

$$\begin{aligned} &\bar{\mu}_0 = \bar{\gamma}_0 (p^0)^{-1} && \text{for} \quad N \text{ odd} \\ &\bar{\mu}_{\pm 1} = \bar{\gamma}_{\pm 1} (p^{\pm 1})^{-1} \mathcal{L}^{\mp 1}{}_1 && \text{for} \quad N \text{ even} \\ &\bar{\mu}_a = \bar{\gamma}_a P_{|a|}^{-1} P_{|a|-1}^{-1} p^{-a} && \text{otherwise} \end{aligned} \tag{5.13}$$

and $\bar{\gamma}_a \in \mathbb{C}$ normalization constants fulfilling the conditions

$$\begin{aligned} &\bar{\gamma}_0 = q^{\frac{1}{2}} h^{-1} && \text{for} \quad N \text{ odd} \\ &\bar{\gamma}_{\pm 1} = k^{-1} && \text{for} \quad N \text{ even} \\ &\bar{\gamma}_1 \bar{\gamma}_{-1} = -q h^{-2} && \text{for} \quad N \text{ odd} \\ &\bar{\gamma}_a \bar{\gamma}_{-a} = -q k^{-2} \omega_a \omega_{a-1} && \text{for} \quad a > 1 \end{aligned} \tag{5.14}$$

From definition (2.12), using (A.13), (5.9), (5.12), (A.14), we find

$$\begin{aligned} \zeta_5^- \left(\mathcal{L}^{-i}{}_j\right) &= \mathcal{L}^{-i}{}_h \tilde{\varphi}^- \left(S\mathcal{L}^{-h}{}_j\right) = \mathcal{L}^{-i}{}_h \left[\mu_h, p^i\right]_q \\ \zeta_5^+ \left(\mathcal{L}^{+i}{}_j\right) &= \mathcal{L}^{+i}{}_h \tilde{\varphi}^+ \left(S\mathcal{L}^{+h}{}_j\right) = \mathcal{L}^{+i}{}_h [\bar{\mu}_h, p^i]_{q^{-1}}. \end{aligned} \tag{5.15}$$

The case of even N is slightly outside the scheme developed in the preceding sections. As anticipated, the generators $\mathcal{L}^{\pm 1}{}_1$ and their inverses $\mathcal{L}^{\pm -1}{}_{-1}$ cannot be realized as (rational) 'functions' of the p's and have to be introduced in the co-domain of $\tilde{\varphi}^{\pm}$ as new generators. In fact definitions (5.9) and (5.12) are exactly designed to lead to

$$\tilde{\varphi}^{\pm} \left(\mathcal{L}^{\pm 1}{}_1\right) = \mathcal{L}^{\pm 1}{}_1 \qquad \tilde{\varphi}^{\pm} \left(\mathcal{L}^{\pm -1}{}_{-1}\right) = \mathcal{L}^{\pm -1}{}_{-1}. \tag{5.16}$$

As a consequence, when $s = \pm 1$

$$\zeta_5^+ \left(\mathcal{L}^{+s}{}_s\right) = \mathcal{L}^{+s}{}_k \tilde{\varphi}^+ \left(S\mathcal{L}^{+k}{}_s\right) \overset{(A.1),(5.16)}{=} \mathcal{L}^{+s}{}_s S\mathcal{L}^{+s}{}_s = 1 \tag{5.17}$$

$$\zeta_5^- \left(\mathcal{L}^{-s}{}_s\right) = \mathcal{L}^{-s}{}_k \tilde{\varphi}^- \left(S\mathcal{L}^{-k}{}_s\right) \overset{(A.2),(5.16)}{=} \mathcal{L}^{-s}{}_s S\mathcal{L}^{-s}{}_s = 1. \tag{5.18}$$

Moreover, it is easy to check that

$$\begin{aligned} \mathcal{L}^{-1}{}_1\zeta_5^{\pm}\left(\mathcal{L}^{\pm i}{}_j\right) &= \zeta_5^{\pm}\left(\mathcal{L}^{\pm i}{}_j\right)\mathcal{L}^{-1}{}_1 q^{\eta_i-\eta_j} \\ \mathcal{L}^{+1}{}_1\zeta_5^{\pm}\left(\mathcal{L}^{\pm i}{}_j\right) &= \zeta_5^{\pm}\left(\mathcal{L}^{\pm i}{}_j\right)\mathcal{L}^{+1}{}_1 q^{\eta_j-\eta_i} \end{aligned} \tag{5.19}$$

where we have introduced the shorthand notation

$$\eta_i := \delta_i^1 - \delta_i^{-1}. \tag{5.20}$$

Therefore, $\zeta_5^{\pm}\big[U_q^{\pm}so(2n)\big]$ do not commute with the whole $\mathcal{A}$ but only with $\mathbb{R}_q^{2n}$, and the decomposition (4.12) will have to be modified into

$$\tilde{\mathcal{C}} = \zeta_5^{+}(H^{+})\zeta_5^{-}(H^{-})\mathcal{Z}\left(\mathbb{R}_q^{2n}\right) = \zeta_5^{-}(H^{-})\zeta_5^{+}(H^{+})\mathcal{Z}\left(\mathbb{R}_q^{2n}\right) \tag{5.21}$$

with $\tilde{\mathcal{C}}$ defined as the commutant of $\mathbb{R}_q^{2n}$ within $\mathbb{R}_q^{2n} \rtimes U_q so(2n)$. As said, $\mathcal{Z}\big(\mathbb{R}_q^{2n}\big)$ also contains $\sqrt{\frac{p^{\pm 1}}{p^{\mp 1}}}$. Since $\mu_{\pm 1}$, $\bar{\mu}_{\mp 1}$ and therefore $\tilde{\varphi}^{-}\left(\mathcal{L}^{-i}{}_{\pm 1}\right)$, $\tilde{\varphi}^{+}\left(\mathcal{L}^{+i}{}_{\mp 1}\right)$ are of the form '$\mathcal{L}^{\mp 1}{}_1 \times$ an expression depending only on the p's', they q-commute rather than commute with $\zeta^{\pm}\big(\mathcal{L}^{\pm i}{}_j\big)$.

For odd and even N the commutation relations among $\zeta_5^{-}\big(\mathcal{L}^{-i}{}_j\big)$ or $\zeta_5^{+}\big(\mathcal{L}^{+i}{}_j\big)$ are immediately obtained from (A.1)–(A.6) and (A.8) applying ζ_5^{-}, ζ_5^{+} and using the fact that they are homomorphisms.

To derive the commutation relations between the $\zeta_5^{-}\big(\mathcal{L}^{-i}{}_j\big)$ and the $\zeta_5^{+}\big(\mathcal{L}^{+h}{}_k\big)$ (the analogue of (2.20) in explicit form) we need the commutation relations between the $\tilde{\varphi}^{-}\big(\mathcal{L}^{-i}{}_j\big)$ and the $\tilde{\varphi}^{+}\big(\mathcal{L}^{+i}{}_j\big)$. To proceed we have to distinguish the case of odd and even N. In the formulation of the following lemma and proposition we switch off the Einstein summation convention. In the appendix we prove

Lemma 1. *If N is odd, then*

$$\tilde{\varphi}^{-}\left(S\mathcal{L}^{-i}{}_k\right)\tilde{\varphi}^{+}\left(S\mathcal{L}^{+h}{}_j\right) = \frac{\gamma_k}{\bar{\gamma}_k}\sum_{l,m,r,s}\hat{R}^{-1}{}^{ih}_{lm}\tilde{\varphi}^{+}\left(S\mathcal{L}^{+l}{}_r\right)\tilde{\varphi}^{-}\left(S\mathcal{L}^{-m}{}_s\right)\hat{R}^{rs}_{kj}\frac{\bar{\gamma}_s}{\gamma_s}. \tag{5.22}$$

If N is even, then if $k \notin \{-1, 1\}$

$$\begin{aligned} \tilde{\varphi}^{-}\left(S\mathcal{L}^{-i}{}_k\right)\tilde{\varphi}^{+}\left(S\mathcal{L}^{+h}{}_j\right) = \frac{\gamma_k}{\bar{\gamma}_k}\sum_{l,m}\hat{R}^{-1}{}^{ih}_{lm}&\left[\sum_{\substack{r,s\\ s\neq\pm 1}}\tilde{\varphi}^{+}\left(S\mathcal{L}^{+l}{}_r\right)\tilde{\varphi}^{-}\left(S\mathcal{L}^{-m}{}_s\right)\hat{R}^{rs}_{kj}\frac{\bar{\gamma}_s}{\gamma_s}\right.\\ &\left.+\sum_{\substack{r,s\\ s=\pm 1}}\tilde{\varphi}^{+}\left(S\mathcal{L}^{+l}{}_r\right)\tilde{\varphi}^{-}\left(S\mathcal{L}^{-m}{}_{-s}\right)\frac{p^{-s}}{p^{s}}\hat{R}^{rs}_{kj}\frac{\bar{\gamma}_s}{\gamma_s}q^2\right] \end{aligned} \tag{5.23}$$

and if $k \in \{-1, 1\}$

$$\begin{aligned} \tilde{\varphi}^{-}\left(S\mathcal{L}^{-i}{}_k\right)\tilde{\varphi}^{+}\left(S\mathcal{L}^{+h}{}_j\right) = -\sum_{l,m}\hat{R}^{-1}{}^{ih}_{lm}&\left[\sum_{\substack{r,s\\ s\neq\pm 1}}\tilde{\varphi}^{+}\left(S\mathcal{L}^{+l}{}_r\right)\tilde{\varphi}^{-}\left(S\mathcal{L}^{-m}{}_s\right)\hat{R}^{rs}_{-k,j}\frac{\bar{\gamma}_s}{\gamma_s}\right.\\ &\left.+\sum_{\substack{r,s\\ s=\pm 1}}\tilde{\varphi}^{+}\left(S\mathcal{L}^{+l}{}_r\right)\tilde{\varphi}^{-}\left(S\mathcal{L}^{-m}{}_{-s}\right)\frac{p^{-s}}{p^{s}}\hat{R}^{rs}_{-k,j}\frac{\bar{\gamma}_s}{\gamma_s}q^2\right]\frac{p^{-k}}{p^{k}}q^{-2+2\eta_j\eta_k}. \end{aligned} \tag{5.24}$$

Note that for odd N if the ratio $\frac{\gamma_a}{\bar{\gamma}_a}$ is independent of a $\big($therefore it must be equal to $\frac{\gamma_0}{\bar{\gamma}_0} = -q^{-1}\big)$, the $\gamma, \bar{\gamma}$'s disappear, and by comparison with (A.7) we find [2] that setting $\tilde{\varphi}\big(\mathcal{L}^{+i}{}_j\big) = \tilde{\varphi}^+\big(\mathcal{L}^{+i}{}_j\big)$, $\tilde{\varphi}\big(\mathcal{L}^{-i}{}_j\big) = \tilde{\varphi}^-\big(\mathcal{L}^{-i}{}_j\big)$ defines a homomorphism $\tilde{\varphi}\colon \mathcal{A}{\rtimes}U_q so(N) \to \mathcal{A}$. For $|q| = 1$, $\tilde{\varphi}$ turns out to be a $*$-homomorphism w.r.t. the corresponding $*$-structures (see below).

Proposition 10. *If N is odd*

$$\zeta_5^+\left(\mathcal{L}^{+h}{}_j\right)\zeta_5^-\left(\mathcal{L}^{-i}{}_k\right) = \sum_{c,d,r,s} \hat{R}^{-1ih}{}_{cd}\zeta_5^-\left(\mathcal{L}^{-d}{}_s\right)\zeta_5^+\left(\mathcal{L}^{+c}{}_r\right)\hat{R}^{rs}_{kj}\frac{\gamma_k}{\bar{\gamma}_k}\frac{\bar{\gamma}_s}{\gamma_s}. \tag{5.25}$$

If N is even, then

$$\begin{aligned}\zeta_5^+\left(\mathcal{L}^{+h}{}_j\right)\zeta_5^-\left(\mathcal{L}^{-i}{}_k\right) = {}& \frac{\gamma_k}{\bar{\gamma}_k}q^{\eta_j(\eta_i-\eta_k)}\sum_{l,m,r}\hat{R}^{-1ih}{}_{lm}\Bigg[\sum_{s\neq\pm1}\zeta_5^-\left(\mathcal{L}^{-m}{}_s\right)\zeta_5^+\left(\mathcal{L}^{+l}{}_r\right)\\ &+\sum_{s=\pm1}\zeta_5^-\left(\mathcal{L}^{-m}{}_{-s}\right)\zeta_5^+\left(\mathcal{L}^{+l}{}_r\right)\frac{p^{-s}}{p^s}q^{2+\eta_s(\eta_r-\eta_l)}\Bigg]\hat{R}^{rs}_{kj}\frac{\bar{\gamma}_s}{\gamma_s}\end{aligned} \tag{5.26}$$

if $k \notin \{1, -1\}$, *and, if* $k \in \{1, -1\}$,

$$\begin{aligned}\zeta_5^+\left(\mathcal{L}^{+h}{}_j\right)\zeta_5^-\left(\mathcal{L}^{-i}{}_k\right) = {}& -q^{-2+\eta_j(\eta_i+\eta_k)}\sum_{l,m,r}\hat{R}^{-1ih}{}_{lm}\Bigg[\sum_{s\neq\pm1}\zeta_5^-\left(\mathcal{L}^{-m}{}_s\right)\zeta_5^+\left(\mathcal{L}^{+l}{}_r\right)\\ &+\sum_{s=\pm1}\zeta_5^-\left(\mathcal{L}^{-m}{}_{-s}\right)\zeta_5^+\left(\mathcal{L}^{+l}{}_r\right)\frac{p^{-s}}{p^s}q^{2+\eta_s(\eta_r-\eta_l)}\Bigg]\zeta_5^+\left(\mathcal{L}^{+l}{}_r\right)\hat{R}^{rs}_{-k,j}\frac{\bar{\gamma}_s}{\gamma_s}\frac{p^{-k}}{p^k}.\end{aligned} \tag{5.27}$$

So for even N the coefficients in the commutation relations depend explicitly on the central $\big($in $\mathbb{R}_q^N\big)$ elements $\frac{p^{\pm1}}{p^{\mp1}}$.

Let us now analyse the properties of $\zeta_5^{\pm}$ under $*$-structures. When $|q| = 1$ the $*$-structure of $\mathbb{R}_q^N$ is given by $(p^i)^* = p^i$. It turns out [9] that $\tilde{\varphi}^{\pm}$ are $*$-homomorphisms if, in addition,

$$\gamma_a^* = -\gamma_a\begin{cases}1 & \text{if} \quad a < -1 \qquad \text{or} \quad a = -1 \text{ and } N \text{ odd}\\ q^{-2} & \text{if} \quad a > 1 \qquad \text{or} \quad a = 1 \text{ and } N \text{ odd.}\end{cases} \tag{5.28}$$

Under these assumptions, in view also of (A.15), we can apply proposition 9 and conclude that $\zeta_5^{\pm}$ are $*$-homomorphisms. When $q \in \mathbb{R}^+$ the real structure of $\mathbb{R}_q^N$ is given by $(p^i)^* = p^j g_{ji}$. It turns out [10] that $\tilde{\varphi}^{\pm}$ fulfil $[\varphi^{\pm}(g)]^* = \varphi^{\mp}(g^*)$, or more explicitly

$$\left[\varphi^-\left(\mathcal{L}^{-i}{}_j\right)\right]^* = \varphi^+\left[\left(\mathcal{L}^{-i}{}_j\right)^*\right] \tag{5.29}$$

if, in addition,

$$\gamma_a^* = -\bar{\gamma}_{-a}\begin{cases}1 & \text{if} \quad a < -1 \qquad \text{or} \quad a = -1 \text{ and } n \text{ odd}\\ q^{-2} & \text{if} \quad a > 1 \qquad \text{or} \quad a = 1 \text{ and } N \text{ odd}\end{cases} \tag{5.30}$$

Under these assumptions, and in view of (A.17) and (A.15) also, we can apply proposition 9 and conclude that $\zeta_5^{\pm}$ fulfil

$$\zeta_5^{\pm}(g^*) = \left[\zeta_5^{\mp}(g)\right]^* \qquad g \in H^{\mp}. \tag{5.31}$$

5.2. The cross-product of $U_q\mathbf{g}$'s with $U_q\mathbf{g}$-covariant Heisenberg algebras

As an algebra $\mathcal{A}$ we shall consider a slight extension of the $U_q\mathbf{g}$-covariant deformed Heisenberg algebras $\mathcal{D}_{\epsilon,\mathbf{g}}$, $\mathbf{g} = sl(N), so(N)$. Such algebras have been introduced in [1, 15, 16]. They are unital associative algebras generated by x^i, ∂_j fulfilling the relations

$$\mathcal{P}_{a\,hk}^{\;ij} x^h x^k = 0 \qquad \mathcal{P}_{a\,hk}^{\;ij} \partial_j \partial_i = 0 \qquad \partial_i x^j = \delta^i_j + (q\gamma\hat{R})^{\epsilon\,jk}_{\;\;ih} x^h \partial_k \tag{5.32}$$

where $\gamma = q^{\frac{1}{N}}$, 1 respectively for $\mathbf{g} = sl(N), so(N)$, and the exponent ϵ can take either value $\epsilon = 1, -1$. $\hat{R}$ denotes the braid matrix of $U_q\mathbf{g}$ (given in formulae (A.9), (A.10)), and the matrix $\mathcal{P}_a$ is the deformed antisymmetric projector appearing in the decompositions (A.11), (A.12) of the latter. The coordinates x^i, as the p^i of section 5.1, transform according to the fundamental N-dimensional representation ρ of $U_q\mathbf{g}$, whereas the 'partial derivatives' transform according to the contragradient representation,

$$x^i \triangleleft g = \rho^i_j(g) x^j \qquad \partial_i \triangleleft g = \partial_h \rho^h_i(S^{-1}g). \tag{5.33}$$

In our conventions the indices will take the values $i = 1, \ldots, N$ if $\mathbf{g} = sl(N)$, whereas if $\mathbf{g} = so(N)$ they will take the same values considered in section 5.1. In fact, the quantum Euclidean space can be considered as a subalgebra of the $U_q so(N)$-covariant Heisenberg algebra, either by the identification $p^i \equiv x^i$ or $p^i \equiv g^{ij}\partial_j$.

Algebra homomorphisms $\tilde{\varphi}: \mathcal{A} \rtimes H \to \mathcal{A}$, for $H = U_q\mathbf{g}$ and $\mathcal{A}$ equal to (a suitable completion of) $\mathcal{D}_{\epsilon,\mathbf{g}}$ have been constructed in [4, 7]. This is the q-analogue of the well-known fact that the elements of $\mathbf{g}$ can be realized as 'vector fields' (first-order differential operators) on the corresponding $\mathbf{g}$-covariant (undeformed) space, e.g. $\tilde{\varphi}(E^i_j) = x^i\partial_j - \frac{1}{N}\delta^i_j$ in the $\mathbf{g} = sl(N)$ case. This means that our decoupling map ζ_5 also exists in the undeformed case; we do not know whether this result in Lie group representation theory has ever been formulated before.

The explicit expression of $\tilde{\varphi}(\mathcal{L}^{-i}{}_j)$ in terms of x^i, ∂_j for $U_q sl(2)$, $U_q so(3)$ has been given in [10]. For different values of N it can be found from the results of [4, 7] by passing from the generators adopted there to the FRT generators. For example, for $\mathbf{g} = sl(2)$ and $\epsilon = 1$ one finds

$$\begin{aligned}
&\tilde{\varphi}\left(\mathcal{L}^{+1}{}_1\right) = \tilde{\varphi}\left(\mathcal{L}^{-2}{}_2\right) = \left[\tilde{\varphi}\left(\mathcal{L}^{-1}{}_1\right)\right]^{-1} = \left[\tilde{\varphi}\left(\mathcal{L}^{+2}{}_2\right)\right]^{-1} = \alpha\Lambda^{\frac{1}{2}}\left[1 + (q^2-1)x^2\partial_2\right]^{\frac{1}{2}}\\
&\tilde{\varphi}\left(\mathcal{L}^{+1}{}_2\right) = -\alpha k q^{-1}\Lambda^{\frac{1}{2}}\left[1 + (q^2-1)x^2\partial_2\right]^{-\frac{1}{2}} x^1\partial_2\\
&\tilde{\varphi}\left(\mathcal{L}^{-2}{}_1\right) = \alpha k q^3\Lambda^{\frac{1}{2}}\left[1 + (q^2-1)x^2\partial_2\right]^{-\frac{1}{2}} x^2\partial_1
\end{aligned} \tag{5.34}$$

where α is fixed by (A.7) to be $\alpha = \pm 1, \pm i$ and we have set

$$\Lambda^{-2} := 1 + (q^2-1)x^i\partial_i \tag{5.35}$$

whereas for $\mathbf{g} = so(3)$ and $\epsilon = 1$ one finds on the positive Borel subalgebra

$$\begin{aligned}
&\tilde{\varphi}\left(\mathcal{L}^{+-}{}_-\right) = -\alpha\Lambda\left[1 + (q-1)x^0\partial_0 + (q^2-1)x^+\partial_+\right]\\
&\tilde{\varphi}\left(\mathcal{L}^{+-}{}_0\right) = \alpha k\Lambda\left(x^-\partial_0 - \sqrt{q}x^0\partial_+\right)\\
&\tilde{\varphi}\left(\mathcal{L}^{+-}{}_+\right) = \tfrac{1}{1+q^{-1}}\tilde{\varphi}\left(\mathcal{L}^{+-}{}_0\right)\tilde{\varphi}\left(\mathcal{L}^{+0}{}_+\right)\\
&\tilde{\varphi}\left(\mathcal{L}^{+0}{}_0\right) = 1\\
&\tilde{\varphi}\left(\mathcal{L}^{+0}{}_+\right) = -q^{-\frac{1}{2}}\left[\tilde{\varphi}\left(\mathcal{L}^{+-}{}_-\right)\right]^{-1}\tilde{\varphi}\left(\mathcal{L}^{+-}{}_0\right)\\
&\tilde{\varphi}\left(\mathcal{L}^{++}{}_+\right) = \left[\tilde{\varphi}\left(\mathcal{L}^{+-}{}_-\right)\right]^{-1}
\end{aligned} \tag{5.36}$$

and on the negative Borel subalgebra

$$\begin{aligned}
&\tilde{\varphi}\left(\mathcal{L}^{-}{}^{-}_{-}\right) = -\left(\alpha\Lambda\left[1+(q-1)x^0\partial_0+(q^2-1)x^+\partial_+\right]\right)^{-1}\\
&\tilde{\varphi}\left(\mathcal{L}^{-}{}^{0}_{-}\right) = -\alpha q^2 k\tilde{\varphi}\left(\mathcal{L}^{-}{}^{-}_{-}\right)\Lambda\left(x^0\partial_- - \sqrt{q}x^+\partial_0\right)\\
&\tilde{\varphi}\left(\mathcal{L}^{-}{}^{+}_{-}\right) = \tfrac{1}{1+q}\tilde{\varphi}\left(\mathcal{L}^{-}{}^{+}_{0}\right)\tilde{\varphi}\left(\mathcal{L}^{-}{}^{0}_{-}\right)\\
&\tilde{\varphi}\left(\mathcal{L}^{-}{}^{0}_{0}\right) = 1\\
&\tilde{\varphi}\left(\mathcal{L}^{-}{}^{+}_{0}\right) = -\alpha q^{\frac{3}{2}}k\Lambda\left(x^0\partial_- - \sqrt{q}x^+\partial_0\right)\\
&\tilde{\varphi}\left(\mathcal{L}^{-}{}^{+}_{+}\right) = \left[\tilde{\varphi}\left(\mathcal{L}^{-}{}^{-}_{-}\right)\right]^{-1}.
\end{aligned} \tag{5.37}$$

Here we have set

$$\Lambda^{-2} := \left[1+(q^2-1)x^i\partial_i + \frac{(q^2-1)^2}{\omega_1^2}\left(g_{ij}x^ix^j\right)\left(g^{hk}\partial_k\partial_h\right)\right] \tag{5.38}$$

where

$$\omega_a := (q^{\rho_a}+q^{-\rho_a})$$

and replaced for simplicity the values $-1, 0, 1$ of the indices by the values $-, 0, +$. In either case the $\tilde{\varphi}$-images of $\mathcal{L}^{+i}_{\ j}$ and $\mathcal{L}^{-j}_{\ \ i}$ for $i > j$ vanish, because the latter do.

We see that strictly speaking $\tilde{\varphi}$ takes values in some appropriate completion of $\mathcal{D}_{\epsilon,\mathbf{g}}$, containing at least the square root and inverse square root of the polynomial Λ^{-2}, respectively, defined in (5.35), (5.38), as well as the square root of $\left[1+(q^2-1)x^2\partial_2\right]$ and its inverse, when $\mathbf{g} = sl(2)$, and the inverses $(5.36)_6$ and $(5.37)_6$, when $\mathbf{g} = so(3)$. Apart from this minimal completion, another possible completion is the so-called h-adic, namely the ring of formal power series in $h = \log q$ with coefficients in $\mathcal{D}_{\epsilon,\mathbf{g}}$. Other completions, e.g. in operator norms, can be considered according to need. One can easily show that the extension of the action of H to any such completion is uniquely determined (we omit to write down its explicit expression, since we do not need it).

$\mathcal{A}\rtimes H$ is a $*$-algebra and the map $\tilde{\varphi}$ is a $*$-homomorphism both for q real and $|q| = 1$. The $*$-structure of $\mathcal{A}$ is

$$(x^i)^* = x^i \qquad (\partial_i)^* = -\partial_i \begin{cases} q^{\pm 2(N-i+1)} & \text{if } H = U_q sl(N) \\ q^{\pm N+\rho_i} & \text{if } H = U_q so(N) \end{cases} \tag{5.39}$$

if $|q| = 1$, and

$$(x^h)^* = x^k g_{kh} \qquad (\partial_i)^* = -\frac{\Lambda^{\pm 2}}{q^{\pm N}+q^{\pm 2}}\left[\left(g^{jh}\partial_h\partial_j\right), x^i\right] \tag{5.40}$$

if $H = U_q so(N)$ and $q \in \mathbb{R}^+$. The upper or lower sign, respectively, refers to the choices $\epsilon = 1, -1$ in $(5.32)_3$, and $\Lambda^{\pm 2}$ are, respectively, defined by

$$\Lambda^{\pm 2} := \left[1+(q^{\pm 2}-1)x^i\partial_i + \frac{(q^{\pm 2}-1)^2}{\omega_n^2}r^2\left(g^{ji}\partial_i\partial_j\right)\right]^{-1}. \tag{5.41}$$

In either case, in view of (A.17) also, we can apply proposition 8 and conclude that ζ_5 is a $*$-homomorphism.

Appendix A. Basic properties of $U_q g$

For both $H = U_q sl(N), U_q so(N)$ the FRT generators fulfil the relations

$$\mathcal{L}^{+i}_{\ \ j} = 0 \qquad \text{if} \quad i > j \tag{A.1}$$

$$\mathcal{L}^{-i}_{\ \ j} = 0 \qquad \text{if} \quad i < j \tag{A.2}$$

$$\mathcal{L}^{-i}{}_{i}\mathcal{L}^{+i}{}_{i} = \mathcal{L}^{+i}{}_{i}\mathcal{L}^{-i}{}_{i} = 1 \qquad \forall i \tag{A.3}$$

$$\prod_i \mathcal{L}^{+i}{}_{i} = 1 \qquad \prod_i \mathcal{L}^{-i}{}_{i} = 1 \tag{A.4}$$

$$\hat{R}^{ab}_{cd}\mathcal{L}^{+d}{}_{f}\mathcal{L}^{+c}{}_{e} = \mathcal{L}^{+b}{}_{c}\mathcal{L}^{+a}{}_{d}\hat{R}^{dc}_{ef} \tag{A.5}$$

$$\hat{R}^{ab}_{cd}\mathcal{L}^{-d}{}_{f}\mathcal{L}^{-c}{}_{e} = \mathcal{L}^{-b}{}_{c}\mathcal{L}^{-a}{}_{d}\hat{R}^{dc}_{ef} \tag{A.6}$$

$$\hat{R}^{ab}_{cd}\mathcal{L}^{+d}{}_{f}\mathcal{L}^{-c}{}_{e} = \mathcal{L}^{-b}{}_{c}\mathcal{L}^{+a}{}_{d}\hat{R}^{dc}_{ef} \tag{A.7}$$

in addition, when $H = U_q so(N)$ they also fulfil

$$\mathcal{L}^{\pm i}{}_{j}\mathcal{L}^{\pm h}{}_{k}g^{kj} = g^{hi} \qquad \mathcal{L}^{\pm j}{}_{i}\mathcal{L}^{\pm k}{}_{h}g_{kj} = g_{hi}. \tag{A.8}$$

Here $\hat{R}^{ab}_{cd}$ denotes the braid matrix of $U_q sl(N)$, $U_q so(N)$, and g^{hi} the metric matrix of $U_q so(N)$, which was defined in (5.7). The square roots of the diagonal elements $\mathcal{L}^{-i}{}_{i}$, as well as the square roots of the diagonal elements $\mathcal{L}^{+i}{}_{i}$, generate the same Cartan subalgebra of H, which coincides with $H^+ \cap H^-$. The braid matrix $\hat{R}$ of $U_q sl(N)$ is given by

$$\hat{R} = q^{-\frac{1}{N}}\left[q\sum_i e^i_i \otimes e^i_i + \sum_{i\neq j} e^j_i \otimes e^i_j + k\sum_{i<j} e^i_i \otimes e^j_j\right] \tag{A.9}$$

where all indices $i, j, a, \ldots = 1, 2, \ldots, N$, and e^i_j is the $N \times N$ matrix with all elements equal to zero except for a 1 in the i column and jth row. When $H = U_q so(N)$ it is convenient to adopt the convention that all indices $i, j, a, \ldots$ take the values $i = -n, \ldots, -1, 0, 1, \ldots, n$ for N odd, and $i = -n, \ldots, -1, 1, \ldots, n$ for N even, where $n := \left[\frac{N}{2}\right]$ is the rank of $so(N)$. Then the corresponding braid matrix reads as

$$\hat{R} = q\sum_{i\neq 0} e^i_i \otimes e^i_i + \sum_{\substack{i\neq j,-j \\ \text{or } i=j=0}} e^j_i \otimes e^i_j + q^{-1}\sum_{i\neq 0} e^{-i}_i \otimes e^i_{-i}$$
$$+k\left(\sum_{i<j} e^i_i \otimes e^j_j - \sum_{i<j} q^{-\rho_i+\rho_j} e^{-j}_i \otimes e^j_{-i}\right). \tag{A.10}$$

The braid matrix of $sl(N)$ admits the orthogonal projector decomposition

$$q^{\frac{1}{N}}\hat{R} = q\mathcal{P}_S - q^{-1}\mathcal{P}_a \qquad \mathbf{g} = sl(N) \tag{A.11}$$

$\mathcal{P}_a, \mathcal{P}_S$ are the $U_q sl(N)$-covariant deformed antisymmetric and symmetric projectors. The braid matrix of $so(N)$ admits the orthogonal projector decomposition

$$\hat{R} = q\mathcal{P}_s - q^{-1}\mathcal{P}_a + q^{1-N}\mathcal{P}_t \qquad \mathbf{g} = so(N) \tag{A.12}$$

$\mathcal{P}_a, \mathcal{P}_t, \mathcal{P}_s$ are the corresponding q-deformed antisymmetric, trace and trace-free symmetric projectors.

By iterated use of equations (A.5)–(A.7) one immediately shows that they also hold if $\hat{R}$ is replaced by any polynomial function $f(\hat{R})$ of $\hat{R}$, in particular by $f(\hat{R}) = \hat{R}^{-1}, \mathcal{P}_a$.

Finally, the co-product of the FRT generators is given by

$$\Delta\left(\mathcal{L}^{+i}{}_{j}\right) = \mathcal{L}^{+i}{}_{h} \otimes \mathcal{L}^{+h}{}_{j} \qquad \Delta\left(\mathcal{L}^{-i}{}_{j}\right) = \mathcal{L}^{+i}{}_{h} \otimes \mathcal{L}^{+h}{}_{j}. \tag{A.13}$$

When $H = U_q so(N)$ the antipode is given by

$$S\mathcal{L}^{\mp j}{}_{i} = g_{ih}\mathcal{L}^{\mp h}{}_{k}g^{kj}. \tag{A.14}$$

The non-compact real sections of $U_g\mathbf{g}$ require $|q| = 1$ and are characterized by the $*$-structure

$$\left(\mathcal{L}^{\pm i}{}_j\right)^* = U^{-1i}{}_r\mathcal{L}^{\pm r}{}_s U^s_j. \tag{A.15}$$

Here one can take

$$U^i_j := \begin{cases} g^{ih}g_{jh} & \text{if} \quad \mathbf{g} = so(N) \\ q^{-i}\delta^i_j & \text{if} \quad \mathbf{g} = sl(N) \end{cases}. \tag{A.16}$$

The compact real section of $U_q\mathbf{g}$ requires $q \in \mathbb{R}^+$ if $\mathbf{g} = so(N)$, $q \in \mathbb{R}$ if $\mathbf{g} = sl(N)$ and is characterized by the $*$-structure

$$\left(\mathcal{L}^{\pm i}{}_j\right)^* = S\mathcal{L}^{\mp j}{}_i. \tag{A.17}$$

For $\mathbf{g} = so(N)$ this amounts to

$$\left(\mathcal{L}^{\pm i}{}_j\right)^* = g_{ih}\mathcal{L}^{\mp h}{}_k g^{kj}. \tag{A.18}$$

Proof of lemma 1

We start by recalling two relations proved in lemma 2 of [2]

$$\begin{aligned} \mu_a\tilde{\varphi}^-\left(S\mathcal{L}^{-i}{}_b\right) &= \hat{R}^{-1cd}_{ab}\tilde{\varphi}^-\left(S\mathcal{L}^{-i}{}_c\right)\mu_d \\ \bar{\mu}_a\tilde{\varphi}^+\left(S\mathcal{L}^{+i}{}_b\right) &= \hat{R}^{cd}_{ab}\tilde{\varphi}^+\left(S\mathcal{L}^{+i}{}_c\right)\bar{\mu}_d. \end{aligned} \tag{A.19}$$

Applying $\tilde{\varphi}^\pm$ to (5.5) we find

$$\tilde{\varphi}^\pm\left(S\mathcal{L}^{\pm a}{}_b\right)p^i = \hat{R}^{\pm1ai}_{jk}p^j\tilde{\varphi}^\pm\left(S\mathcal{L}^{\pm k}{}_b\right). \tag{A.20}$$

We finally note that

$$\bar{\mu}_{\pm1} = -\mu_{\mp1}\frac{p^{\mp1}}{p^{\pm1}}q^2. \tag{A.21}$$

The claim is a direct consequence of relations (A.19), (A.20) and (A.21). □

Proof of proposition 10

For odd N

$$\begin{aligned}
\text{lhs}(5.25) &\overset{(5.15)}{=} \sum_a \mathcal{L}^{+h}{}_a\tilde{\varphi}^+\left(S\mathcal{L}^{+a}{}_j\right)\zeta_5^-\left(\mathcal{L}^{-i}{}_k\right) \\
&\overset{\text{Thm 4}}{=} \sum_a \mathcal{L}^{+h}{}_a\zeta_5^-\left(\mathcal{L}^{-i}{}_k\right)\tilde{\varphi}^+\left(S\mathcal{L}^{+a}{}_j\right) \\
&\overset{(5.15)}{=} \sum_{a,b} \mathcal{L}^{+h}{}_a\mathcal{L}^{-i}{}_b\tilde{\varphi}^-\left(S\mathcal{L}^{-b}{}_k\right)\tilde{\varphi}^+\left(S\mathcal{L}^{+a}{}_j\right) \\
&\overset{(5.22)}{=} \sum_{a,b}\sum_{l,m,r,s} \mathcal{L}^{+h}{}_a\mathcal{L}^{-i}{}_b\hat{R}^{-1ba}_{lm}\tilde{\varphi}^+\left(S\mathcal{L}^{+l}{}_r\right)\tilde{\varphi}^-\left(S\mathcal{L}^{-m}{}_s\right)\hat{R}^{rs}_{kj}\frac{\gamma_k}{\bar{\gamma}_k}\frac{\bar{\gamma}_s}{\gamma_s} \\
&\overset{(A.7)}{=} \sum_{c,d}\sum_{l,m,r,s} \hat{R}^{-1ih}_{cd}\mathcal{L}^{-d}{}_m\mathcal{L}^{+c}{}_l\tilde{\varphi}^+\left(S\mathcal{L}^{+l}{}_r\right)\tilde{\varphi}^-\left(S\mathcal{L}^{-m}{}_s\right)\hat{R}^{rs}_{kj}\frac{\gamma_k}{\bar{\gamma}_k}\frac{\bar{\gamma}_s}{\gamma_s} \\
&\overset{(5.15)}{=} \sum_{c,d}\sum_{m,r,s} \hat{R}^{-1ih}_{cd}\mathcal{L}^{-d}{}_m\zeta_5^+\left(\mathcal{L}^{+c}{}_r\right)\tilde{\varphi}^-\left(S\mathcal{L}^{-m}{}_s\right)\hat{R}^{rs}_{kj}\frac{\gamma_k}{\bar{\gamma}_k}\frac{\bar{\gamma}_s}{\gamma_s} \\
&\overset{\text{Thm 4}}{=} \sum_{c,d}\sum_{m,r,s} \hat{R}^{-1ih}_{cd}\mathcal{L}^{-d}{}_m\tilde{\varphi}^-\left(S\mathcal{L}^{-m}{}_s\right)\zeta_5^+\left(\mathcal{L}^{+c}{}_r\right)\hat{R}^{rs}_{kj}\frac{\gamma_k}{\bar{\gamma}_k}\frac{\bar{\gamma}_s}{\gamma_s} \\
&\overset{(5.15)}{=} \text{rhs}(5.25).
\end{aligned}$$

For even N

$$\begin{aligned}
\zeta_5^+\left(\mathcal{L}^{+h}{}_j\right)\zeta_5^-\left(\mathcal{L}^{-i}{}_k\right) &\overset{(5.15)}{=} \sum_b \mathcal{L}^{+h}{}_b\tilde{\varphi}^+\left(S\mathcal{L}^{+b}{}_j\right)\zeta_5^-\left(\mathcal{L}^{-i}{}_k\right)\\
&\overset{\text{Thm 4},(5.19)}{=} \sum_b \mathcal{L}^{+h}{}_b\zeta_5^-\left(\mathcal{L}^{-i}{}_k\right)\tilde{\varphi}^+\left(S\mathcal{L}^{+b}{}_j\right)q^{\eta_j(\eta_i-\eta_k)}\\
&\overset{(5.15)}{=} \sum_{a,b} \mathcal{L}^{+h}{}_b\mathcal{L}^{-i}{}_a\tilde{\varphi}^-\left(S\mathcal{L}^{-a}{}_k\right)\varphi^+\left(S\mathcal{L}^{+b}{}_j\right)q^{\eta_j(\eta_i-\eta_k)}\\
&\overset{(A.7)}{=} \sum_{a,b}\sum_{l,m,c,d} \hat{R}^{-1}{}^{ih}_{lm}\mathcal{L}^{-m}{}_d\mathcal{L}^{+l}{}_c\hat{R}^{cd}_{ab}\tilde{\varphi}^-\left(S\mathcal{L}^{-a}{}_k\right)\varphi^+\left(S\mathcal{L}^{+b}{}_j\right)q^{\eta_j(\eta_i-\eta_k)}.
\end{aligned}$$

If $k \notin \{1, -1\}$ then

$$\begin{aligned}
\zeta_5^+\left(\mathcal{L}^{+h}{}_j\right)\zeta_5^-\left(\mathcal{L}^{-i}{}_k\right) &\overset{(5.23)}{=} \frac{\gamma_k}{\bar{\gamma}_k}\sum_{l,m,c,d}\hat{R}^{-1}{}^{ih}_{lm}\mathcal{L}^{-m}{}_d\mathcal{L}^{+l}{}_c\Bigg[\sum_{\substack{r,s\\ s\neq\pm1}}\tilde{\varphi}^+\left(S\mathcal{L}^{+c}{}_r\right)\\
&\quad\times\tilde{\varphi}^-\left(S\mathcal{L}^{-d}{}_s\right)\hat{R}^{rs}_{kj}\frac{\bar{\gamma}_s}{\gamma_s} + \sum_{\substack{r,s\\ s=\pm1}}\tilde{\varphi}^+\left(S\mathcal{L}^{+c}{}_r\right)\tilde{\varphi}^-\left(S\mathcal{L}^{-d}{}_{-s}\right)\frac{p^{-s}}{p^s}\hat{R}^{rs}_{kj}\frac{\bar{\gamma}_s}{\gamma_s}q^2\Bigg]q^{\eta_j(\eta_i-\eta_k)}\\
&\overset{(5.15)}{=} \frac{\gamma_k}{\bar{\gamma}_k}\sum_{l,m,d,r}\hat{R}^{-1}{}^{ih}_{lm}\mathcal{L}^{-m}{}_d\zeta_5^+\left(\mathcal{L}^{+l}{}_r\right)\Bigg[\sum_{s\neq\pm1}\tilde{\varphi}^-\left(S\mathcal{L}^{-d}{}_s\right)\hat{R}^{rs}_{kj}\frac{\bar{\gamma}_s}{\gamma_s}\\
&\quad+\sum_{s=\pm1}\tilde{\varphi}^-\left(S\mathcal{L}^{-d}{}_{-s}\right)\frac{p^{-s}}{p^s}\hat{R}^{rs}_{kj}\frac{\bar{\gamma}_s}{\gamma_s}q^2\Bigg]q^{\eta_j(\eta_i-\eta_k)}\\
&\overset{\text{Thm 4},(5.19)}{=} \frac{\gamma_k}{\bar{\gamma}_k}\sum_{l,m,d,r}\hat{R}^{-1}{}^{ih}_{lm}\mathcal{L}^{-m}{}_d\Bigg[\sum_{s\neq\pm1}\tilde{\varphi}^-\left(S\mathcal{L}^{-d}{}_s\right)\hat{R}^{rs}_{kj}\frac{\bar{\gamma}_s}{\gamma_s}\\
&\quad+\sum_{s=\pm1}\tilde{\varphi}^-\left(S\mathcal{L}^{-d}{}_{-s}\right)\frac{p^{-s}}{p^s}\hat{R}^{rs}_{kj}\frac{\bar{\gamma}_s}{\gamma_s}q^{2+\eta_s(\eta_r-\eta_l)}\Bigg]\zeta_5^+\left(\mathcal{L}^{+l}{}_r\right)q^{\eta_j(\eta_i-\eta_k)}\\
&\overset{(5.15)}{=} \frac{\gamma_k}{\bar{\gamma}_k}\sum_{l,m,r}\hat{R}^{-1}{}^{ih}_{lm}\Bigg[\sum_{s\neq\pm1}\zeta_5^-\left(\mathcal{L}^{-m}{}_s\right)\hat{R}^{rs}_{kj}\frac{\bar{\gamma}_s}{\gamma_s}\\
&\quad+\sum_{s=\pm1}\zeta_5^-\left(\mathcal{L}^{-m}{}_{-s}\right)\frac{p^{-s}}{p^s}\hat{R}^{rs}_{kj}\frac{\bar{\gamma}_s}{\gamma_s}q^{2+\eta_s(\eta_r-\eta_l)}\Bigg]\zeta_5^+\left(\mathcal{L}^{+l}{}_r\right)q^{\eta_j(\eta_i-\eta_k)}\\
&\overset{\text{Thm 4}}{=} \text{rhs}(5.26).
\end{aligned}$$

If $k \in \{1, -1\}$ then

$$\zeta_5^+\left(\mathcal{L}^{+h}{}_j\right)\zeta_5^-\left(\mathcal{L}^{-i}{}_k\right) \stackrel{(5.23)}{=} -\sum_{\substack{l,m,c\\d,r}} \hat{R}^{-1}{}^{ih}_{lm}\mathcal{L}^{-m}{}_d\mathcal{L}^{+l}{}_c\tilde{\varphi}^+\left(S\mathcal{L}^{+c}{}_r\right)\left[\sum_{s\neq\pm1}\tilde{\varphi}^-\left(S\mathcal{L}^{-d}{}_s\right)\right.$$

$$\left.\times\frac{\bar{\gamma}_s}{\gamma_s}\hat{R}^{rs}_{-k,j} + \sum_{\substack{s\\s=\pm1}}\tilde{\varphi}^-\left(S\mathcal{L}^{-d}{}_{-s}\right)\frac{p^{-s}}{p^s}\hat{R}^{rs}_{-kj}\frac{\bar{\gamma}_s}{\gamma_s}q^2\right]q^{\eta_j(\eta_i-\eta_k)}\frac{p^{-k}}{p^k}q^{-2+2\eta_j\eta_k}$$

$$\stackrel{(5.15)}{=} -\sum_{l,m,d,r}\hat{R}^{-1}{}^{ih}_{lm}\mathcal{L}^{-m}{}_d\zeta_5^+(\mathcal{L}^{+l}{}_r)\left[\sum_{s\neq\pm1}\tilde{\varphi}^-\left(S\mathcal{L}^{-d}{}_s\right)\hat{R}^{rs}_{-k,j}\frac{\bar{\gamma}_s}{\gamma_s}\right.$$

$$\left.+\sum_{s=\pm1}\tilde{\varphi}^-\left(S\mathcal{L}^{-d}{}_{-s}\right)\frac{p^{-s}}{p^s}\hat{R}^{rs}_{-k,j}\frac{\bar{\gamma}_s}{\gamma_s}q^2\right]q^{-2+\eta_j(\eta_i+\eta_k)}\frac{p^{-k}}{p^k}$$

$$\stackrel{\text{Thm 4,(5.19)}}{=} -\sum_{l,m,r}\hat{R}^{-1}{}^{ih}_{lm}\mathcal{L}^{-m}{}_d\left[\sum_{s\neq\pm1}\tilde{\varphi}^-\left(S\mathcal{L}^{-d}{}_s\right)\hat{R}^{rs}_{-k,j}\frac{\bar{\gamma}_s}{\gamma_s}\right.$$

$$\left.+\sum_{s=\pm1}\tilde{\varphi}^-\left(S\mathcal{L}^{-d}{}_{-s}\right)\frac{p^{-s}}{p^s}\hat{R}^{rs}_{-k,j}\frac{\bar{\gamma}_s}{\gamma_s}q^{2+\eta_s(\eta_r-\eta_l)}\right]\zeta_5^+\left(\mathcal{L}^{+l}{}_r\right)q^{-2+\eta_j(\eta_i+\eta_k)}\frac{p^{-k}}{p^k}$$

$$\stackrel{(5.15)}{=} -\sum_{l,m,r}\hat{R}^{-1}{}^{ih}_{lm}\left[\sum_{s\neq\pm1}\zeta_5^-\left(\mathcal{L}^{-m}{}_s\right)\hat{R}^{rs}_{-k,j}\frac{\bar{\gamma}_s}{\gamma_s}+\sum_{s=\pm1}\zeta_5^-\left(\mathcal{L}^{-m}{}_{-s}\right)\right.$$

$$\left.\times\frac{p^{-s}}{p^s}\hat{R}^{rs}_{-k,j}\frac{\bar{\gamma}_s}{\gamma_s}q^{2+\eta_s(\eta_r-\eta_l)}\right]\zeta_5^+\left(\mathcal{L}^{+l}{}_r\right)q^{-2+\eta_j(\eta_i+\eta_k)}\frac{p^{-k}}{p^k}$$

$$\stackrel{\text{Thm 4}}{=} \text{rhs}(5.27).$$

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